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Logarithmic Spirals and Insects

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Logarithmic Spirals and Insects

By

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Abstract

The logarithmic spiral, also known as the growth spiral, is an interesting form in mathematics that happens to be very applicable to the natural world. We explore the structure of this curve, and how this spiral can be used to model the flight pattern of a moth. Specifically, we investigate assumptions used to construct models for insect flight. Analysis of these underlying assumptions gives insight into possible improvements to and limitations of specific models.

1. Introduction

Ever noticed a moth at night, how they mindlessly spiral porch lights and street lamps? Well this spiral may not be quite as mindless as it seems. In fact there is a very specific reason for the flight path of certain insects. The spiral is a result of an adaptation gone wrong. Before the existence of artificial light, moths developed a helpful adaptation in order to fly straight. A moth has a compound eye structure that has a part called the Ommatidia. This part of the eye will stay activated by a light ray if it flies at a constant angle to the light source. When the moth only had the light rays of the sun and moon, this adaptation allowed them to fly a perfectly straight path, but because artificial light is much closer in proximity to the moth, the adaptation to keep the Ommatidia activated causes them to fly in a spiral rather than a straight path. More specifically the path that is formed is a geometric pattern called a logarithmic spiral. A mathematician

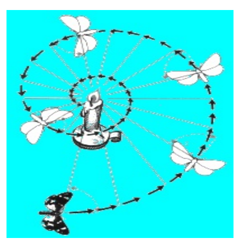


Figure 1: A Moth Circling a Candle Light

named Khristo Boyadzhiev created a model of this moth phenomenon [3]. We can start by analyzing his model so that we can continue his work and get a better understanding of moth flight.

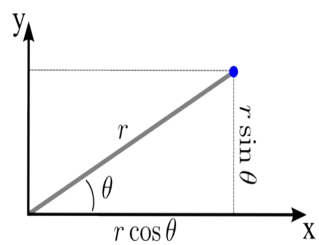
2. Background

In order to fully understand modeling a moth's flight pattern around a light we will review some of the basic mathematics that go into building a model.

2.1. Polar Coordinates

Switching back and forth from polar coordinates to rectangular coordinates can often be helpful in expressing behavior, especially in this case when we are dealing with a circular motion. These type of coordinates can make an equation more legible or sometimes simplify an equation. It is helpful to recall the following.

Theorem 2.1 *If P is a point with polar coordinates (r, θ), the rectangular coordinates (x,y) of P are given by*



$$x = r\cos(\theta), \text{ and } y = r\sin(\theta)$$

While rectangular coordinates may be more familiar to some readers, polar coordinates are useful because they take into account distance from the origin and angle measurement from the polar axis.

2.2. Vectors

A moth's flight pattern will express a direction and also magnitude, therefore vectors will be very useful when describing insect behavior. Vectors have some specific rules and applications. When working with these vectors we often will be using real numbers(scalars) and vectors in correspondence with each other. Because we will be working so closely with vectors, there is a couple of theorems and definitions that will be important to be familiar with.

Definition 2.2 *If α is a scalar and $\bar{v}$ is a vector, the scalar product $\alpha\bar{v}$ is defined as follows:*
1. If $\alpha > 0$ the product $\alpha\bar{v}$ is the vector whose magnitude is α times the magnitude of $\bar{v}$ and whose direction is the same as $\bar{v}$.
2. If $\alpha < 0$, the product $\alpha\bar{v}$ is the vector whose magnitude is $|\alpha|$ times the magnitude of $\bar{v}$ and whose direction is opposite that of $\bar{v}$.
3. If $\alpha = 0$ or if $\bar{v} = 0$ then $\alpha\bar{v} = 0$.

Furthermore, let $||\bar{V}||$ be the notation for the magnitude of a vector $\bar{V}$, where $||\bar{V}||$ is the norm of $\bar{V}$. The norm is simply the length of a directed line segment.

Theorem 2.3 *Properties of $||\bar{V}||$*
If V is a vector and if α is a scalar, then
a.) $||\bar{V}|| \geq 0$ b.) $||\bar{V}|| = 0$ iff $V = 0$
c.) $||-\bar{V}|| = ||\bar{V}||$ d.) $||\alpha\bar{V}|| = |\alpha|||\bar{V}||$

By taking the norm of a vector we are able to describe the magnitude and direction of the moths flight.

In the case that we are dealing with multiple vectors, there is another concept to consider. That concept would be angle measure between vectors. In our case, where we will be trying to keep a constant angle between vectors, angle measure will be important. The dot product gives us a way to describe this angle.

Definition 2.4 *The* ***dot product*** *of two nonzero vectors $\bar{a}$ and $\bar{b}$ is the number:*

$$\bar{a} \cdot \bar{b} = |\bar{a}||\bar{b}| \cos(\theta),$$

where θ is the angle between $\bar{a}$ and $\bar{b}$, and $0 \leq \theta \leq \pi$. (So θ is the smaller angle between the vectors when they are drawn with the same initial point.) If either $\bar{a}$ or $\bar{b}$ is 0, we define $\bar{a} \cdot \bar{b} = 0$.

Building on the concept of a dot product our last concept to consider before we begin our model is tangent vectors. A tangent vector is created from a position vector, much like the position vectors described previously. For example if we have a position vector say $\bar{p}(t) =< x(t), y(t) >$, where x(t) and y(t) are just regular functions with respect to time, then the tangent vector would simply be $\bar{p'}(t) =< x'(t), y'(t) >$. This tangent vector describes the line tangent to the trajectory at any given time. By definition trajectory is the path of a projectile. Since a moth can be considered a projectile, trajectory will be an useful thing to consider.

Keeping these ideas in mind we can begin our model. Just like in our background where we built upon simple ideas, our model will also begin with the basics.

3. Building a 2D Model

The first model we will find is a basic 2D model for a moths flight around a light. To really understand this model it helps to start with some simple concepts. First off, our model will be graphed on the Cartesian plane. We will center our light on our coordinate plane by letting the origin represent the light source position.

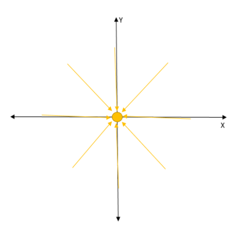


Figure 2: The Basic Cartesian plane

Then if we think of the light rays as vectors we can let $\bar{x}$ represent the vector directed toward the light. Furthermore, we will also consider the tangent vector to the light vector, and call this $\bar{x'}$. This tangent vector will be the moth's flight direction and is pictured below.

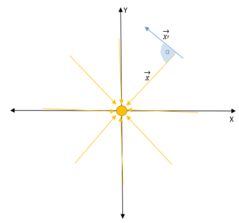


Figure 3: Visuals of $\bar{x}$ and $\bar{x'}$

Now that we have the basic concepts, we must take into consideration one of the defining aspects of our model.

One of the most important pieces to this model is the constant angle. Let α represent this angle. It is important to note that not any value for α will produce a spiral. We can further specify α by observing some natural constraints. In order to create a spiral we must have our constant angle measure between $0 \leq \alpha \leq \pi$. When our constant angle is within these parameters, the angle is sharp enough so that the moth is turning but broad enough so that it doesn't fly directly at the light. The only way for the moth to fly straight is if $\alpha = 0$ (flying straight at the light)

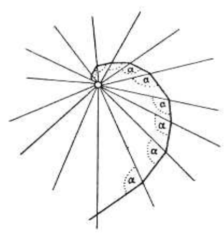


Figure 4: Spiral visual

or $\alpha = \pi$ (flying straight away from the light).When $\alpha = \frac{\pi}{2}$ then a perfect circle is obtained. We will only be concerned with the cases for α that create a spiral.
If we assume that the light is centered at the origin and the position vector for the insect is $\bar{x} =< x(t), y(t) >$, then we can start with the parameterized polar coordinates,

$$x(t) = r(t)\cos(\theta(t)), y(t) = r(t)\sin(\theta(t)) \tag{3.1}$$

In the above equations $x(t)$ is the horizontal component of $\bar{x} =< x(t), y(t) >$ and $y(t)$ is the vertical component at time t. One might notice that these equations are a little different than the polar coordinates defined above. The difference is that these equations are slightly more complex because they consider the time t.
If we differentiate the above equations we can use chain rule on 3.1 to get:

$$\begin{aligned} x'(t) &= r'\cos(\theta) - r\sin(\theta)\theta' \text{ and,} \\ y'(t) &= r'\sin(\theta) + r\cos(\theta)\theta' \end{aligned} \tag{3.2}$$

These equation will come in handy later in our model. We will use these when looking at the tangent vector.

Moving forward we have another influential equation to consider. The logarithmic spiral mimics the flight pattern of these moths quite well due to the fact that both spirals have a constant angle. Referring to [6] a logarithmic spiral can be represented by,

$$r(t) = ae^{b\theta(t)}$$

Again, notice that we have enhanced this equation by looking at it with respect to time.
We can also use the parametric equations for the logarithmic spiral which again referring to [6] are,

$$x(t) = r(t)\cos(\theta) \text{ and } y(t) = r(t)\sin(\theta)$$

With the simple substitution of r(t) we then have the equations

$$x(t) = ae^{b\theta(t)}\cos(\theta(t)) \text{ and } y(t) = ae^{b\theta(t)}\sin(\theta(t))$$

One thing of particular interest to us in this model is the tangent vector at every point. Recall that the tangent vector represents the moth's flight direction. To find the tangent vector we must derive the parametric vector $\bar{x} =< x, y >$. To do this we can use our equations from 3.2 to get

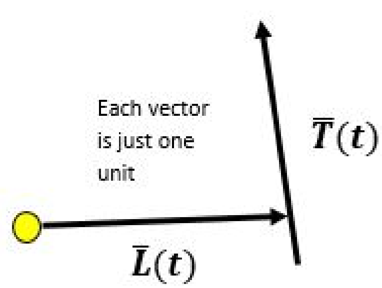
$$\begin{aligned}\bar{x'} &= \frac{d}{dt} < r(t)\cos(\theta(t)), r(t)\sin(\theta(t)) > \\ &=< r'\cos(\theta) - r\sin(\theta)\theta', r'\sin(\theta) + r\cos(\theta)\theta' >\end{aligned}$$

Now there are some other vectors we want to consider. Let the unit light ray vector be denoted by $\bar{L}$ and the tangent unit vector be $\bar{T}$. Specifically we have,

$$\bar{T}(t) = \frac{\bar{x'}}{||\bar{x'}||} = \frac{< x', y' >}{\sqrt{(x')^2 + (y')^2}} \tag{3.3}$$

$$\bar{L}(t) = \frac{- < x, y >}{||- < x, y > ||} \tag{3.4}$$

These vectors have an important relationship in relation to the vectors we previously discussed, $\bar{x}$ and $'\bar{x}$. The vector $\bar{T}$ is just the unit version of our tangent vector $\bar{x'}$ and the vector $\bar{L}$ is simply the vector that is going the opposite direction of $\bar{x}$.



What is most important about these vectors, is their relationship. We again are referring to the constant angle between these vectors because for our model to describe moth flight, the angle between these vectors must not change. In vector calculus the angle between two vectors can be described by the dot product of the vectors. Therefore, we want to consider the following product:

$$\bar{T}(t) \cdot \bar{L}(t) = ||\bar{T}(t)|| \cdot ||\bar{L}(t)|| \cos(\alpha)$$

or

$$\cos(\alpha) = \frac{\bar{T}(t) \cdot \bar{L}(t)}{||\bar{T}(t)|| \cdot ||\bar{L}(t)||}.$$

Since were dealing with two unit vectors and the norm is the length, we have

$$||\bar{T}(t)|| \cdot ||\bar{L}(t)|| = 1 \text{ and therefore } \cos(\alpha) = \bar{T}(t) \cdot \bar{L}(t).$$

Furthermore by substitution of 3.3 and 3.4 we get that,

$$\begin{aligned}
\cos(\alpha) &= \frac{\bar{x'}(t)}{||\bar{x'}(t)||} \cdot \frac{- < x(t), y(t) >}{|| < x(t), y(t) > ||} \\
&= \frac{-(x'(t)x(t) + y'(t)y(t))}{\sqrt{(x'(t))^2 + (y'(t))^2}\sqrt{x(t)^2 + y(t)^2}},
\end{aligned}$$

and in polar coordinates our equation becomes

$$\begin{aligned}
-\cos(\alpha) &= \frac{r\cos(\theta')(r'\cos(\theta) - r\sin(\theta)\theta') + r\sin(\theta)(r'\sin(\theta) + r\cos(\theta)\theta')}{\sqrt{(r')^2 + (\theta')^2 r^2}} \\
&= \frac{r'\cos^2(\theta) - r\cos(\theta)\sin(\theta)\theta' + r'\sin^2(\theta) + r\sin(\theta)\cos(\theta)\theta'}{\sqrt{(r')^2 + (\theta')^2 r^2}} \\
&= \frac{r'(\cos^2(\theta) + \sin^2(\theta))}{\sqrt{(r')^2 + (\theta')^2 r^2}} \\
&= \frac{r'}{\sqrt{(r')^2 + (\theta')^2 r^2}}.
\end{aligned}$$

If we put this all together we have the following equation, that is sometimes referred to as the general differential equation of equal angular motion.

$$\frac{x(t)x'(t) + y(t)y'(t)}{\sqrt{x(t)^2 + y(t)^2}\sqrt{(x'(t))^2 + (y'(t))^2}} = \frac{r'(t)}{\sqrt{(r'(t))^2 + (\theta'(t))^2 r(t)^2}} = -\cos(\alpha) \tag{3.5}$$

We will be referring back to this equation several times. Now we will make the natural assumption that an insect has a constant speed. This assumption gives us

$$\|\bar{x'}\|^2 = (x'(t))^2 + (y'(t))^2 = (r'(t))^2 + (\theta'(t))^2(r(t))^2 = v^2 = \mathit{constant}. \tag{3.6}$$

We can refer back to our general differential equation for equal-angular motion 3.5 which gave us, $\frac{r'(t)}{\sqrt{r'(t)^2+\theta'(t)^2(r(t))^2}} = -\cos(\alpha)$ and substitute in 3.6.

$$\frac{r'(t)}{\sqrt{v^2}} = \frac{r'(t)}{v} = -\cos(\alpha)$$

Multiplying both sides by v gets $r'(t) = -v\cos(\alpha)$. Since we want to find $r(t)$ (distance from the origin with respect to time) we can integrate with $r_0 = r(0)$ to finally get,

$$r(t) = \int -\cos(\alpha) = -(v\cos(\alpha))t + r_0 \tag{3.7}$$

This gives us $r(t)$, therefore we have half of the information we need. Now we just want to find $\theta(t)$ so that we can graph our spiral in polar coordinates. Since we aim to find $\theta(t)$ we will substitute $r'(t) = -v\cos(x)$ into 3.6. In 3.6 we have $(r'(t))^2 + (\theta'(t))^2 r(t)^2 = v^2$, thus by substitution of $r'(t)$ we get that

$$v^2 = (-v\cos(\alpha))^2 + \theta'(t)^2 r(t)^2 = v^2\cos^2(\alpha) + \theta'(t)^2 r(t)^2.$$

In order to solve for $\theta(t)$ we must first solve for $\theta'(t)$. In order to do this we will use a combination of factoring, trig identities, square roots and division to obtain the following:

$$\begin{aligned} v^2 - v^2\cos^2(\alpha) &= \theta'(t)^2 r(t)^2 \\ v^2(1-\cos^2(\alpha)) &= \theta'(t)^2 r(t)^2 \\ v^2\sin^2(\alpha) &= \theta'(t)^2 r^2(t) \\ v\sin(\alpha) &= \theta'(t) r(t) \end{aligned}$$

Finally, dividing by r gives us our equation, $\frac{v\sin(\alpha)}{r(t)} = \theta'(t)$. Since our goal is to find $\theta(t)$ we can integrate $\theta'(t)$ and substitute in 3.7 to get:

$$\theta(t) = \int \frac{v\sin(\alpha)}{r(t)}dt = \int \frac{v\sin(\alpha)}{r_0 - (v\cos(\alpha))t}dt$$

Through the use of substitution with $u = r_0 - (v\cos(\alpha))t$ and $\frac{du}{dt} = -v\cos(\alpha)$ we have:

$$vdt = -\frac{-1}{\cos(\alpha)}du$$

Substituting this into our integral yields the equation

$$\theta(t) = \int \frac{1}{u}\frac{-\sin(\alpha)}{\cos(\alpha)}du = -\tan(\alpha)\int \frac{1}{u}du$$

in which we finally get

$$\theta(t) = -\tan(\alpha) - \ln(r_0 - v\cos(\alpha)t) + c,$$

where c is a constant of integration. Note that in the equation above we need $r_0 > vt\cos(\alpha)$, which means $0 \leq t < \frac{r_0}{v\cos(\alpha)}$. This is the natural time interval of the flight.

Lastly, if we set $\theta_0 = \theta(0)$ we get,

$$\theta(t) = \theta_0 - \tan(\alpha)\ln(r_0 - vt\cos(\alpha)) = \theta_0 - \ln\left(1 - \frac{vt}{r_0}\cos(\alpha)\right)\tan(\alpha) \tag{3.8}$$

This gives us our model in polar coordinates. Although polar coordinates are sufficient, we can still express $r(t)$ and $\theta(t)$ in new ways. Equations in rectangular coordinates may be more familiar to some audiences.

Converting back to rectangular coordinates using 3.8 and 3.7 we have

$$\begin{aligned} x(t) &= (-(v\cos(\alpha))t + r_0)\cos\theta(t) \\ &= (r_0 - v\cos(\alpha))\cos(\theta(t)) \\ &= (r_0 - vt\cos(\alpha))\sin\left(\theta_0 - \ln\left(1 - \frac{vt}{r_0}\right)\cos(\alpha)\tan(\alpha)\right) \end{aligned}$$

and

$$\begin{aligned} y(t) &= (-v\cos(\alpha)t + r_0)\sin(\theta(t)) \\ &= (r_0 - v\cos(\alpha))\sin(\theta(t)) \\ &= (r_0 - vt\cos(\alpha))\sin\left(\theta_0 - \ln\left(1 - \frac{vt}{r_0}\cos(\alpha)\right)\tan(\alpha)\right). \end{aligned}$$

Additionally, there is another way we can express this flight. We can aim to get the trajectory of the flight to be of the form

$$r(\theta(t)), -\infty < \theta < \infty.$$

In order to find this equation we eliminate t from 3.8 and 3.7. Solving for t in 3.7 yields:

$$t = -\frac{r(t) - r_0}{v\cos(\alpha)}$$

We then may plug t into 3.8 to get the following:

$$\begin{aligned}
\theta(t) &= \theta_0 - \ln\left(1 + \frac{\frac{v(r-r_0)}{-v\cos(\alpha)}}{r_0}\cos(\alpha)\right)\tan(\alpha) \\
&= \theta_0 - \ln\left(1 + \frac{1}{r_0}\frac{v(r-r_0)}{-v\cos(\alpha)}\cos(\alpha)\right)\tan(\alpha) \\
&= \theta_0 - \ln\left(1 + \frac{(r-r_0)}{r_0}\right)\tan(\alpha) \\
&= \theta_0 - \ln\left(1 + \frac{(r-r_0)}{r_0}\right)\tan(\alpha) \\
&= \theta_0 - \ln\left(\frac{r}{r_0}\right)\tan(\alpha)
\end{aligned}$$

If we simplify the above equation and solve for $r(\theta(t))$ we get,

$$r(\theta(t)) = r_0 e^{\cot(\alpha)(\theta_0 - \theta(t))} \tag{3.9}$$

Our model now can be expressed visually in a graph. We can create a graph using our parametric equations(doesn't matter what equations we use) to get a few examples of some possible flight paths of moths.

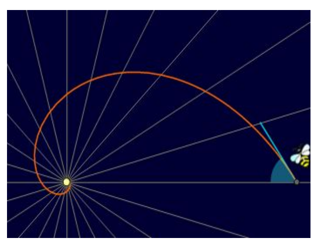


Figure 5: 2D graph of constant speed when $\alpha = 55°$

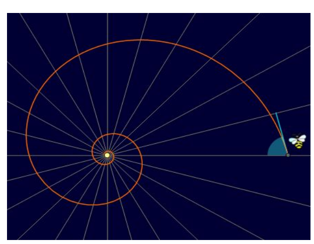


Figure 6: 2D graph of constant speed when $\alpha = 75°$

Now that we have a basic model we must ask ourselves if it captures the behavior of the insects that we observe in nature. We can find that there are some discrepancies between this model and the observable behavior of an insect. Some possible discrepancies with a model such as this is that while it works efficiently for a small insect(small inertia) and small velocity, with larger insects such as bees, this model will not be as accurate due to distortions. Also notice the flight will end at $t = \frac{r_0}{v\cos(\alpha)}$ but 3.8 requires an infinite amount of turns around the light source because,

$$-\ln\left(1 - \frac{vt}{r_0}\cos(\alpha)\right) \to +\infty \text{ when } t \to \frac{r_0}{v\cos(\alpha)}$$

While this model has some small critiques, what is interesting is the assumptions. This model does a great job at communicating controlled conditions of moth flight but it fails to take into account some of natures variables. One assumption that is particularly interesting about this model, is the assumption of constant speed. A moth's speed may in fact vary as it flies, rather than stay constant. If we are considering the inconsistencies of nature then it only makes sense to explore variable speed.

3.1. Hypothesis: Variable Speed

To begin this process we must first discuss the possible ways that a moth's velocity may vary. My gut reaction to this spiral was to compare it to the human world. When a human runs they tend to slow as they turn for accuracy's sake. They run slower as the turns get tighter. It is possible that a moths flight may be similar. This brings us to our first consideration for speed. Quite possibly the moth may slow down as they tighten their spiral around the light. This can be represented by a simple equation such as this

$$||\bar{v}(t)|| = v - \epsilon t \text{ or (in the case it spirals out) } ||\bar{v}(t)|| = v + \epsilon t$$

where t is our time, v is some constant and ϵ is a very small number. The nice thing about this varied speed is that it builds eloquently off our previous velocity assumption. For example if we let $\epsilon = 0$ then we will be back to our previous model. For this model we will be assuming that ϵ is a small number. Let $0 < \epsilon < .5$. We constrict ϵ to allow for the speed change to be very slight. This is a simple way to describe the speed in more detail.

To understand how to alter this model we must first recall our original 2D model. We can build our model using some of our previous models work. Conveniently enough, everything before 3.6 remains unchanged. Since we can use part of our old model we will follow a similar process to what was done to create our 2D constant speed model.

Since our equations for varied speed are quite similar, with out loss of generality we will be attempting to incorporate the first equation into our model. Therefore the speed will be

$$||\bar{v}(t)||^2 = (x'(t))^2 + (y'(t))^2 = (r'(t))^2 + (\theta'(t))^2 r(t)^2 = (v - \epsilon t)^2. \tag{3.10}$$

Building on our model at where we assumed velocity, we can use the general differential equation for equal-angular motion (3.5) to get:

$$r'(t) = -(v - \epsilon t)\cos(\alpha) \text{ or } r'(t) = (-v + \epsilon t)\cos(\alpha) = -v\cos(\alpha) + \epsilon t\cos(\alpha)$$

Quickly note that when we square $r'(t)$ we obtain:

$$\begin{aligned}(r'(t))^2 &= (-v\cos(\alpha)+\epsilon t\cos(\alpha))(-v\cos(\alpha)+\epsilon t\cos(\alpha)) \\ &= v^2\cos^2(\alpha)-2\epsilon vt\cos^2(\alpha)+\epsilon^2t^2\cos^2(\alpha) \\ &= \cos^2(\alpha)(v^2-2\epsilon vt+\epsilon^2t^2) \\ &= \cos^2(\alpha)(v-\epsilon t)^2\end{aligned} \tag{3.11}$$

Putting $(r'(t))^2$ to the side for a second, we can again let $r_0 = r(0)$ which allows us to integrate $r'(t)$ to get,

$$\begin{aligned}r(t) &= \int -(v-\epsilon t)\cos(\alpha)dt \\ &= \int -v\cos(\alpha)+\epsilon\cos(\alpha)tdt \\ &= -v\cos(\alpha)t+\frac{\epsilon\cos(\alpha)t^2}{2}+r_0\end{aligned} \tag{3.12}$$

We now have the first part to our parametric polar coordinate equations. The next step is to find $\theta(t)$ so that we may graph our spiral. To do this we will refer back to our varied velocity 3.10, and substitute in 3.11:

$$\begin{aligned}(v-\epsilon t)^2 &= \cos^2(\alpha)(-v+\epsilon t)^2+(\theta'(t))^2r^2(t) \\ &= \cos^2(\alpha)(v^2-2\epsilon vt+\epsilon^2t^2)+(\theta'(t))^2r^2(t)\end{aligned}$$

Then solving for $\theta'(t)$ we get

$$(\theta'(t))^2 = \frac{(v-\epsilon t)^2-\cos^2(\alpha)(v^2-2\epsilon vt+\epsilon^2t^2)}{r^2(t)}$$

and

$$\theta'(t) = \sqrt{\frac{(v-\epsilon t)^2-\cos^2(\alpha)(v^2-2\epsilon vt+\epsilon^2t^2)}{r^2(t)}}.$$

Our goal is to find the equation for the position vector $\theta(t)$, so we will want to integrate:

$$\theta(t) = \int\left(\sqrt{\frac{(v+\epsilon t)^2-\cos^2(\alpha)(v^2-2\epsilon vt+\epsilon^2t^2)}{r^2(t)}}\right)dt \tag{3.13}$$

Since we are integrating it helps to find $r^2(t)$ and substitute it in to our denominator. If we square $r(t)$ we get

$$r^2(t) = \left(-v\cos(\alpha)t+\frac{\epsilon\cos(\alpha)t^2}{2}+r_0\right)^2$$

which substitutes in $\theta(t)$ to give us the integral

$$\theta(t) = \int \left(\sqrt{\frac{(v-\epsilon t)^2 - \cos^2(\alpha)(v^2 - 2\epsilon v t + \epsilon^2 t^2)}{\left(-v\cos(\alpha)t + \frac{\epsilon\cos(\alpha)t^2}{2} + r_0\right)^2}} \right) dt. \tag{3.14}$$

Above we can see that we are faced with a challenging integral. Surprisingly enough, this equation is in fact integrable. While it is a integrable, the resulting antiderivative is a bit of a mess. Rather than deal with a complicated function, we can use MatLab to evaluate this integral. For our purposes, we used the built-in function **ode45** to generate $\theta(t)$ for various values of the parameters in our model.

For the first graph of our model we have set $\alpha = 55°$ and $v = 7.33fps$(feet per second). These numbers are based off some brief research on moths. Moths can travel at greater speeds but for this model we are assuming the moth to be flying an average speed. Next for this model we let $\epsilon = 0.2$. This will match our hypothesis because it will only create a slight variation in speed like we predicted. Setting $\epsilon = 0.2$ creates a variation of only $0.6fps$. The spiral will only last for about $3s$. We also have to consider our starting point. If we let $r_0 = 5ft$ it allows us enough distance for the spiral to form. Lastly, there is one more very interesting thing we have done. These graphs not only plot our new model but also plot our old one as well. If we look closely, we can see there is two lines on each graph.

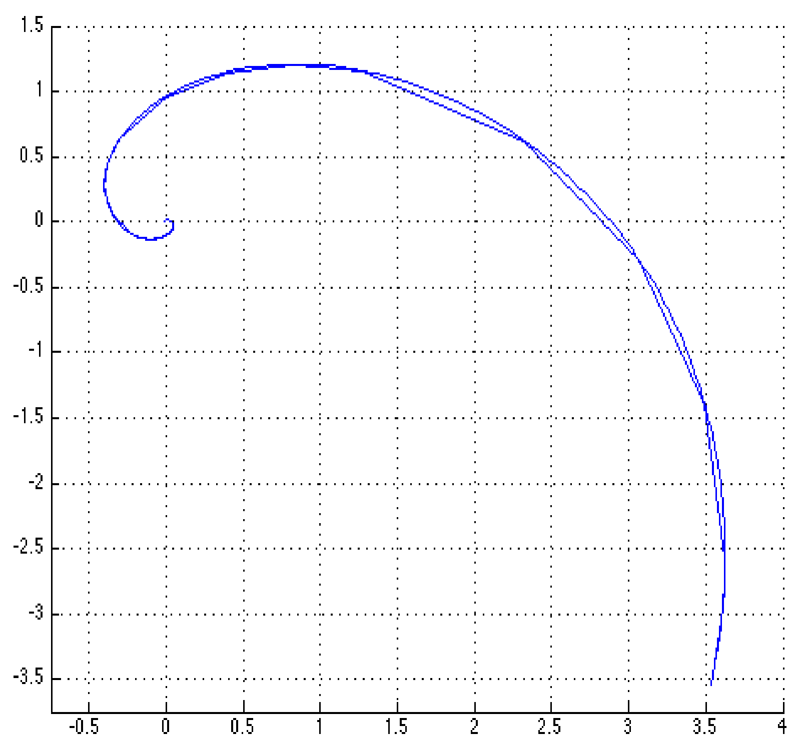


Figure 7: Comparison between constant and varied speed when $\alpha = 55°$

Our second graph only differs in that it has a different angle for α. This time $\alpha = 75°$.

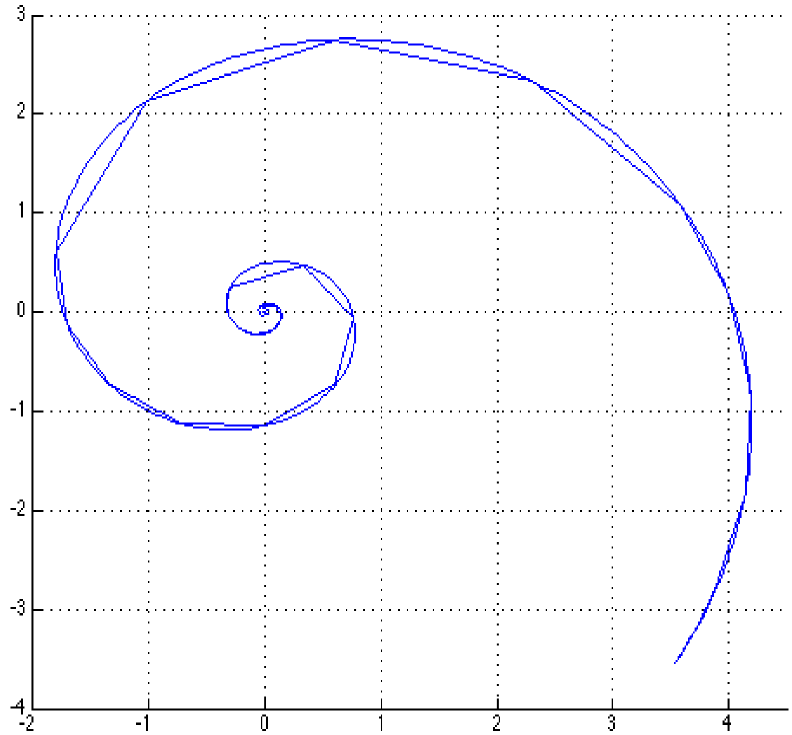


Figure 8: Comparison between constant and varied speed when $\alpha = 75°$

We can see above that our new model is very similar to our old one. Starting with analyzing our graphs visually we observe that our new model is almost exactly the same as our previous one, no matter what the angle is. This suggests that varied speed does not significantly change our graph. The moth is going to fly a similar flight path even when it's speed is changed. Thus in the 2D world our model is not going change by much. This leads me to believe that Boyadzhiev's assumption of constant speed was a good assumption. Through testing the hypothesis we realize that changing to varied speed creates more challenging equations that produce the same overall results. Because of this, depending on how someone is using this model, varied speed is not necessarily considered an improvement.

While is apparent that this model may not be as beneficial as predicted, there are some small advantages we should review. If we dive one step further past the visual of our model, we will see that while the graphs look similar, the data tells a different story. If we look at this model purely from a data stand point, we will see some changes in our new model. The path will remain the same but the time when the moth is traveling that path will change. For example, when $\alpha = 75°$ in the old model the flight stops at $t = 2.6$ but in the new one it doesn't end till $t = 2.7$. That is a 3.8% increase in t between the models. Similarly there is also a difference in the graph where $\alpha = 55°$. In these data sets the old model ended at $t = 1.18$ but the new didn't end until $t = 1.21$. Here we have as much as a 2.5% increase. With these numbers, we can tell that our new model does give us some new information. The advantage with using this new model(varied speed) is we can set ϵ to any small number we would like. It is convenient because ϵ allows us to tailor our model to different moth speeds. Overall though, if one is not looking to predict to the millisecond where a moth will be on a flight path, then the old model with constant speed is much more practical. The results of our new model suggests that speed will not have a dire effect on the basic concepts of the old model.

Looking further ahead it would be interesting to try out some different equations for varied speed. While our equation didn't seem to have a large effect, that does not mean that no versions of varied speed will. This are lots of other ways that a insects could vary their speed. Different insects have different behavior when it come to flight. A larger insect such as a bee may be very inconsistent in its speed. It may speed up and slow down. To account for the fluctuating speed, another hypothesis we could try out in the future would be to let:

$$||\bar{v}|| = v + \epsilon \sin(wt)$$

It would be interesting to see if this varied speed would again create a model similar to our constant speed or if this time we would find more variations. There are many approaches to the speed that we could consider. The research has no bounds, which brings us to another aspect of this model. Boyadzhiev did not stop in the 2D world.

3.2. 3D Flight

We can dive further into this models by considering the 3D model with constant speed. We can summarize Boyadzhiev's work:
Now that we will be working with 3 coordinates (cylindrical) let

$$x = r\cos(\theta), y = r\sin(\theta), z = z$$

Assuming a constant angle α we get that

$$\frac{\bar{x}\bar{x'}}{||\bar{x}||||\bar{x'}||} = \frac{r'(t)r(t) + z(t)z'(t)}{\sqrt{r^2(t) + z^2(t)}\sqrt{(r'(t))^2 + (\theta'(t))^2 r^2(t) + (z'(t))^2}} = -\cos(\alpha). \tag{3.15}$$

Above we have obtained the general differential equation for motion with 3 dimensions instead of 2. To create this equation we used 3.5 from our 2D model. Even though we are in 3D, α is still the angle between the vector $-\bar{x}$ and $\bar{x'}$, however we now have another angle to consider. Let β be the constant angle between radius vector $\bar{x}$ (the direction of the light) and the positive z-axis.
We can let $\bar{k}$ represent the unit vector on the positive vertical axis in which $\bar{k} = (0, 0, 1)$.

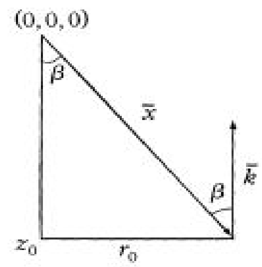


Figure 9: Our vectors for 3D

If $z < 0$ then:

$$\frac{\bar{k} \cdot \bar{x}}{||\bar{k}|| \cdot ||\bar{x}||} = \frac{z}{\sqrt{r^2 + z^2}} = -\cos(\beta)$$

Next, squaring the above equation yields,

$$\frac{z^2}{r^2 + z^2} = \cos^2(\beta)$$

and

$$\frac{r^2 + z^2}{z^2} = \frac{1}{\cos^2(\beta)}$$

Solving for $r^2(t)$ we get:

$$\frac{z^2(t) + r^2(t)}{z^2(t)} = \frac{1}{\cos^2(\beta)}$$

$$r^2(t) = \frac{z^2(t)}{\cos^2(\beta)} - z^2(t)$$

$$r^2(t) = z^2(t) \left(\frac{1}{\cos^2(\beta)} - 1 \right),$$

$$r^2(t) = z^2(t) \tan^2(\beta) \quad \text{or} \quad x^2(t) + y^2(t) = (\tan^2(\beta)) z^2(t) \tag{3.16}$$

Then since the trajectory lies on a circular cone centered at (0,0,0) and opening at 2β we find

$$r^2(t) + z^2(t) = \frac{z^2(t)}{\cos^2(\beta)}$$

which has the derivative of

$$r(t)r'(t) + z(t)z'(t) = \frac{z(t)z'(t)}{\cos^2(\beta)}$$

We can substitute $r(t)r'(t) + z(t)z'(t)$ in 3.15 with $\bar{k} = (0, 0, 1)$ to get that

$$\frac{r(t)r'(t) + z(t)z'(t)}{\sqrt{r^2(t) + z^2(t)}\sqrt{(r'(t))^2 + (\theta'(t))^2 r(t)^2 + (z'(t))^2}} = \frac{z'(t)}{\sqrt{(r'(t))^2 + r(t)^2 + (z'(t))^2}} = \cos(\alpha)\cos(\beta) \tag{3.17}$$

Conveniently this can also be written as:

$$\frac{\bar{k} \cdot \bar{x'}}{||\bar{k}||||\bar{x'}||} = \cos(\alpha)\cos(\beta).$$

Next if we let $\cos(\lambda) = \cos(\alpha)\cos(\beta)$, we can further simplify this equation to be

$$\frac{\bar{k}\bar{x'}}{||\bar{k}||||\bar{x'}||} = \cos(\lambda)$$

This allows us to better understand what we are modeling. By simplifying our equation we can now see that $\bar{x'} = (x', y', z')$ keeps a constant angle with the vector $\bar{k}$ in a horizontal direction.

Diverting our attention back to 3.16, by taking the square root of both sides of the equation shows that,

$$r(t) = -z(t)\tan(\beta)$$

(Note that $\sqrt{z} = -z$ because $z < 0$.)

which derives to be

$$r'(t) = -z'(t)\tan(\beta)$$

Since by 3.16, $r^2(t) + z^2(t) = \frac{z^2(t)}{\cos^2(\beta)}$ then

$$(r'(t))^2 + (z'(t))^2 = \frac{(z'(t))^2}{\cos^2(\beta)}.$$

This continues to simplify:

$$\begin{aligned}
(r'(t))^2 + (z'(t))^2 + r^2(t) &= \frac{(z'(t))^2}{\cos^2(\beta)} + r^2(t) \\
&= \frac{(z'(t))^2}{\cos^2(\beta)} + \frac{r^2(t)\cos^2(\beta)}{\cos^2(\beta)} \\
&= \frac{(z'(t))^2 + r^2(t)\cos^2(\beta)}{\cos^2(\beta)} \\
&= \frac{(z')^2 + z\sin^2(\beta)}{\cos^2(\beta)}
\end{aligned}$$

If we use this equation in our General Differential equation for motion by substituting it in to 3.17 then

$$\cos(\beta)\cos(\alpha) = \frac{z'}{\sqrt{(r')^2 + r^2 + (z')^2}} = \frac{z'}{\sqrt{\frac{(z')^2 + z^2\sin^2(\beta)}{\cos^2(\beta)}}}$$

Now we divide by $\cos(\beta)$ on both sides of the equation:

$$\cos(\alpha) = \frac{z'}{\sqrt{(z')^2 + z^2\sin^2(\beta)}} \text{ equivalent to } \frac{z^2\sin^2(\beta) + (z')^2}{(z')^2} = \frac{1}{\cos^2(\alpha)}$$

Simplification then leads us to,

$$\begin{aligned} \frac{1}{\cos^2(\alpha)} &= \frac{(z')^2\left(1 + \frac{z^2}{(z')^2}\sin^2(\beta)\right)}{(z')^2} \\ &= 1 + \frac{z^2}{(z')^2}\sin^2(\beta) \\ &= 1 + \frac{z^2}{(z')^2}\sin^2(\beta) \end{aligned}$$

Which gives,

$$\frac{z^2}{(z')^2}\sin^2(\beta) = \frac{1}{\cos^2(\alpha)} - 1$$

then,

$$\frac{z^2}{(z')^2}\sin^2(\beta) = \frac{1 - \cos^2(\alpha)}{\cos^2(\alpha)}$$

and finally,

$$\frac{z^2}{(z')^2}\sin^2(\beta) = \tan^2(\alpha).$$

Now that we have an equation, we can notice that there are several ways to express it. Just four of these ways are:

$$\frac{z'}{z} = -\frac{\sin(\beta)}{\tan(\alpha)} \ , \ \frac{dz}{z} = \sin(\beta)\cot(\alpha)d\theta \ , \ \ln(|z|) = -\sin(\beta)\cot(\alpha\theta) + C \text{ and, } z = z_0 e^{-(\sin(\beta)\cot(\alpha))\theta}$$

Since we previously had the equation for rate to be $r = -z\tan(\beta)$. We can substitute in our above equation for z(the last equation in the list) to get:

$$r = -z\tan(\beta) = -\tan(\beta)z_0 e^{-(\sin(\beta)\cot(\alpha))\theta}$$

Finally if we set our initial rate $r_0 = -\tan(\beta)z_0$ and $m = \sin(\beta)\cot(\alpha)$ we get the equations for trajectory(curves that describe the movement with respect to time) to be:

$$x = r\cos(\theta) = r_0 e^{-m\theta}\cos(\theta)$$

$$y = r\sin(\theta) = r_0 e^{-m\theta}\sin(\theta)$$
$$z = z_0 e^{-m\theta}$$

Similarly to our first 2D models we will again assume constant speed.

$$v = ||\bar{x'}|| = \sqrt{(r')^2 + (r)^2 + (z')^2}$$

Recall that in Euclidean space the norm of x would be:

$$||x|| := \sqrt{x_1^2 + ... + x_n^2}$$

We can use this assumed constant speed in combination with equation 3.17 to get,

$$z' = \frac{dz}{dt} = v\cos(\alpha)\cos(\beta)$$

The integral of z' with respect to time would be

$$z = (v\cos(\alpha)\cos(\beta))t + r_0 \tag{3.18}$$

Then since we previously solved for the rate(r) we can substitute in our new formula for z to get that

$$r(t) = -z(t)\tan(\beta) = -(v\cos(\alpha)\sin(\beta))t + r_0 \tag{3.19}$$

Referring back to our assumed constant speed and norm we can square both sides of the equation for v to get the equation

$$v^2 = (r'(t))^2 + (\theta'(t))^2 r^2(t) + (z'(t))^2$$

Now, by substituting in $r'(t)$ and $z'(t)$, we get

$$v^2\cos(\alpha) + (\theta'(t))^2 r(t)^2 = v^2 \text{ or } \theta'(t) = \frac{v\sin(\alpha)}{r(t)}$$

If we substitute in 3.19 and integrate $\theta'(t)$ we get

$$\theta(t) = \int \frac{v\sin(\alpha)dt}{r_0 - (v\cos(\alpha)\sin(\beta))t} = -\frac{\tan(\alpha)}{\sin(\beta)}\ln\left(1 + \frac{vt}{z_0}\cos(\alpha)\cos(\beta)\right) + \theta_0. \tag{3.20}$$

(Note: we have factored out r_0 which is equivalent to $r_0 = -z_0\tan(\beta)$). Finally, we want to get to our law of motion which requires several pieces of information combined. To describe this motion we must use our parameter equations for x,y, and z along with the equations 3.18, 3.19 and 3.20. For simplicity sake it offers clarity to let $a = v\cos(\alpha)\cos(\beta)$, $b = v\cos(\alpha)\sin(\beta)$ and $c = \frac{\tan(\alpha)}{\sin(\beta)}$. We get the following polar coordinate equations.

$$x = (r_0 - bt)\cos\left(\theta_0 - c\ln(1 + \frac{at}{z_0})\right)$$
$$y = (r_0 - bt)\sin\left(\theta_0 - c\ln(1 + \frac{at}{z_0})\right)$$
$$z = z_0 + at$$

Here are a few examples of the 3D spirals with constant speed:

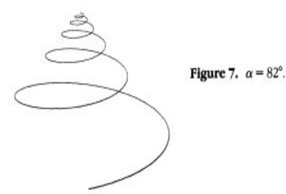


Figure 10: 3D Constant Speed Model when $\alpha = 82°$

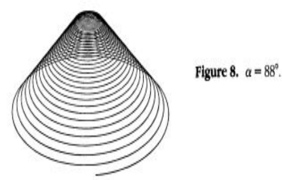


Figure 11: 3D Constant Speed Model when $\alpha = 82°$

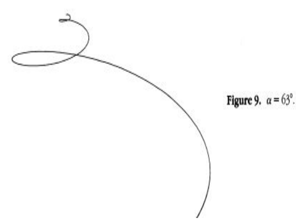


Figure 12: 3D Constant Speed Model when $\alpha = 63°$

This gives us a 3D model. The 3D model has many advantages. One very apparent advantage is the visual product. It is easy to see the moth's flight path when viewing this 3D model. When the model is put in it's natural 3D dimensions it makes it seem more realistic. However, there are not as many advantages to the 3D model as one would think. Since we are dealing with a logarithmic spiral there is no overlap of coordinates on our 2D graph. This makes the graph very easy to read in 2-dimensions. Not only that but the trajectory and position all remains virtually the same. Although we didn't find it necessary to vary the speed in our 3D model, the 3D version is still very important because it has a potential to be altered in the future. As for now though, we are satisfied with the understanding of the 3D model with assumed constant speed.

3.3. Conclusion

Overall, a lot was accomplished during this rebuild of a mathematics model. An interesting article has been broken down to the very last detail and the structure of Boyzshiev's model has been completely analyzed. We created a new varied speed model and analyzed the constant speed 3D and 2D models. The purpose of taking so much time to analyze every step of our model was so that we could truly understand the constant speed model and its underlying mathematics. This bring us to our first product.

The first product of this project is the structure. The break down of the constant speed 2D and 3D model gives a platform for anyone who wants to conduct future research. In a way we have translated previous works into a lower level understanding so that anyone interested can continue our expansion on moth flight. This break down will hopefully make our article accessible to larger audiences.

The next product of our research is the valuable information we received from testing our hypothesis. We predicted that varied speed would be an improvement on our model. However, our results suggested that our hypothesis is not true. While this may seem like a loss, this information gives us something valuable. It verifies that the previous model's assumption of constant speed was accurate. These findings not only give the previous model credibility but also allow for a new level of confidence when graphing these flight patterns in 2D. With this confidence we can move on to explore more aspects of the moths flight.

In the future we have several things to consider. First and foremost, it would be fascinating to consider the 3D version to see if it changes when we vary the speed. Conveniently enough, since we have broken down the mathematics in Boyadzhiev's model we have a platform that we can build a new 3D model off of if we choose to. This 3D model will hopefully help us to further understand the effects of varied speed. Next, we also have different variations to try. Lager insects have yet to be taken into consideration. It may be practical to create a model that would work for all shapes and sizes of insects. This gives us yet another possibility for our future models. Clearly, there is a future for this mathematics research. This model is just a stepping stone for further mathematics discoveries. Flight patterns are one more way that math can continue to broaden our understanding of the natural world.

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