Classifying Regular Polytopes in Dimension 4 and Beyond

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Motivating Questions

How many regular convex polytopes are there in each dimension?
How can we prove this?
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- How can we prove this?
Definition

A **polygon** is a closed and connected shaped bounded by a finite number of lines.

**Polygons**

**Not Polygons**
Regular Convex Polygons

Definition

A **regular convex polygon** is a polygon that is equilateral, equiangular, and whose interior forms a convex set.

Regular Convex Polygons

![Regular Convex Polygons](image)

Not Regular Convex Polygons

![Not Regular Convex Polygons](image)
Regular Convex Polygons

Important things to notice:

- All sides are the same lengths
- The interior angles are all congruent
- They can be represented by a Schlafli symbol
  
  This is of the form \(\{p\}\)
  
  where \(p\) represents the number of sides
  
  It is unique
  
  \(\{3\}\)
  
  \(\{4\}\)
  
  \(\{5\}\)
  
  \(\{6\}\)
  
  \(\{7\}\)
  
  (we can keep going!)
  
  There are infinitely many!

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Classifying Regular Polytopes in Dimension 4 and Beyond
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{3} \quad {4} \quad {5} \quad {6} \quad {7}

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Regular Convex Polygons

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\[
\begin{align*}
\{3\} & \quad \{4\} & \quad \{5\} & \quad \{6\} & \quad \{7\}
\end{align*}
\]

(we can keep going!)

- There are infinitely many!
QUESTION: What is the 3-dimensional analog of the 2-dimensional regular convex polygons?
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ANSWER: The regular convex polyhedra, or the **Platonic solids**
Regular Convex Polyhedra (The Platonic Solids)

Square

Cube

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Classifying Regular Polytopes in Dimension 4 and Beyond
Regular Convex Polyhedra (The Platonic Solids)

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- **Cube**
  - All faces are congruent
Regular Convex Polyhedra (The Platonic Solids)

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  - Notice that the faces are a regular polygon!
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- Interior angles are congruent

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- All faces are congruent
  - Notice that the faces are a regular polygon!
- Angles formed by faces are congruent
Regular Convex Polyhedra (The Platonic Solids)

\[ \{ p, q \} \]
Regular Convex Polyhedra (The Platonic Solids)

\[
\{ p, q \} \\
\{ 4, 3 \}
\]
Regular Convex Polyhedra (The Platonic Solids)

\{ p, q \}

\{4, 3\}

Square facet

\{4\}
Regular Convex Polyhedra (The Platonic Solids)

\[ \{ p, q \} \]

\[ \{ 4, 3 \} \]

Square facet

Triangle vertex figure
Regular Convex Polyhedra (The Platonic Solids)

Definition
An \( n \)-polytope’s **facets** are the type of \((n - 1)\)-polytopes that bound it.

Definition
The **vertex figure** of an \( n \)-polytope is the \((n - 1)\)-dimensional convex hull formed by connecting the center of each of the \((n - 2)\)-elements that are incident on a given vertex.
Regular Convex Polyhedra (The Platonic Solids)

**Theorem**

For a regular convex polyhedron \( \{p, q\} \), \( q\phi < 2\pi \) where \( \phi \) is the interior angle of a regular \( p \)-gon.
Regular Convex Polyhedra (The Platonic Solids)

Theorem

For a regular convex polyhedron \( \{p, q\} \), \( q\phi < 2\pi \) where \( \phi \) is the interior angle of a regular \( p \)-gon.

In other words, the sum of the interior angles that meet at a given vertex must be less than \( 2\pi \).
Regular Convex Polyhedra (The Platonic Solids)

**Theorem**

There are exactly 5 Platonic Solids.
Regular Convex Polyhedra (The Platonic Solids)

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Proof (Outline):
Regular Convex Polyhedra (The Platonic Solids)

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*There are exactly 5 Platonic Solids.*

**Proof (Outline):**

The Platonic Solids have a Schlafli symbol of the form \( \{p, q\} \).

Recall that \( p, q \geq 3 \).
Regular Convex Polyhedra (The Platonic Solids)

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2. Find the interior angle \( \phi \) of a regular \( p \)-gon
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1. Pick a value for \( p \geq 3 \)
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3. Find the values for \( q \geq 3 \) that satisfy \( q\phi < 2\pi \)
Regular Convex Polyhedra (The Platonic Solids)

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3. Find the values for \( q \geq 3 \) that satisfy \( q\phi < 2\pi \)

This will tell us which combinations of \( p \) and \( q \) will give us Platonic solids.
Regular Convex Polyhedra (The Platonic Solids)

Example:

Let $p = 3$, the interior angle of a regular 3-gon (i.e. an equilateral triangle) is $\phi = \frac{\pi}{3}$. This means that we're looking for $q$ that satisfy

$$q \left( \frac{\pi}{3} \right) < 2\pi.$$ 

3. $\left( \frac{\pi}{3} \right) = \frac{\pi}{3} < 2\pi$.

4. $\left( \frac{\pi}{3} \right) = \frac{\pi}{3} < 2\pi$.

5. $\left( \frac{\pi}{3} \right) = \frac{\pi}{3} < 2\pi$.

6. $\left( \frac{\pi}{3} \right) = \frac{\pi}{3} < 2\pi$.

$q = \{3, 4, 5\}$ is satisfied by $q \in \{3, 4, 5\}$.

Therefore, $\{3, 3\}, \{3, 4\}, \{3, 5\}$ are all Platonic solids!
Regular Convex Polyhedra (The Platonic Solids)

Example:

1. Suppose $p = 3$

\[
\begin{align*}
\phi &= \frac{\pi}{3} \\
\phi \left( \frac{\pi}{3} \right) &= \pi < 2\pi \\
\phi \left( \frac{\pi}{3} \right) &= 1 < 2\pi \\
\phi \left( \frac{\pi}{3} \right) &= 2\pi \frac{\pi}{3} < 2\pi
\end{align*}
\]

This means that $\{3,3\}, \{3,4\}, \{3,5\}$ are all Platonic solids!
Regular Convex Polyhedra (The Platonic Solids)

Example:

1. Suppose $p = 3$
2. The interior angle of a regular 3-gon (i.e. an equilateral triangle) is $\phi = \frac{\pi}{3}$.

\[ \phi = \frac{\pi}{3} \]
Example:

1. Suppose \( p = 3 \)

2. The interior angle of a regular \( 3 \)-gon (i.e. an equilateral triangle) is \( \phi = \frac{\pi}{3} \). This means that we’re looking for \( q \) that satisfy \( q \left( \frac{\pi}{3} \right) < 2\pi \).
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$q \left( \frac{\pi}{3} \right) < 2\pi$ is satisfied by $q \in \{3, 4, 5\}$

\[\phi = \frac{\pi}{3}\]
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6 \left( \frac{\pi}{3} \right) = 2\pi
\]

3. \( q \left( \frac{\pi}{3} \right) < 2\pi \) is satisfied by \( q \in \{3, 4, 5\} \)

This means that \( \{3, 3\} \), \( \{3, 4\} \), and \( \{3, 5\} \) are all Platonic solids!
Regular Convex Polyhedra (The Platonic Solids)

If we do this for $p = 4$, we get that $q$ can only be 3.
Regular Convex Polyhedra (The Platonic Solids)

If we do this for $p = 4$, we get that $q$ can only be 3.

If we do this for $p = 5$, we get that $q$ can only be 3.
Regular Convex Polyhedra (The Platonic Solids)

If we do this for \( p = 4 \), we get that \( q \) can only be 3.

If we do this for \( p = 5 \), we get that \( q \) can only be 3.

If we do this for \( p \geq 6 \), we don’t get any \( q \geq 3 \) that work (this is because the interior angle of these polygons is so large).
Regular Convex Polyhedra (The Platonic Solids)

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If we do this for \( p = 5 \), we get that \( q \) can only be 3.

If we do this for \( p \geq 6 \), we don’t get any \( q \geq 3 \) that work (this is because the interior angle of these polygons is so large).

Therefore there are exactly five Platonic solids, given by

\[
\{3, 3\} \quad \{3, 4\} \quad \{3, 5\} \quad \{4, 3\} \quad \{5, 3\}
\]
# Regular Convex Polyhedra (The Platonic Solids)

<table>
<thead>
<tr>
<th>Name of Solid</th>
<th>Faces</th>
<th>Schlafli symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>tetrahedron</td>
<td>4</td>
<td>{3, 3}</td>
</tr>
<tr>
<td>octahedron</td>
<td>8</td>
<td>{3, 4}</td>
</tr>
<tr>
<td>icosahedron</td>
<td>20</td>
<td>{3, 5}</td>
</tr>
<tr>
<td>cube</td>
<td>6</td>
<td>{4, 3}</td>
</tr>
<tr>
<td>dodecahedron</td>
<td>12</td>
<td>{5, 3}</td>
</tr>
</tbody>
</table>
Regular Convex Polyhedra (The Platonic Solids)

- **Square**
  - Bounded by lines
  - All sides are congruent
  - Interior angles are congruent

- **Cube**
  - Bounded by regular polygons
  - All faces are congruent
  - Angles formed by faces are congruent

- **Tesseract**
  - Bounded by regular polyhedra
  - All cells are congruent
  - Angles formed by cells are congruent

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Regular Convex Polyhedra (The Platonic Solids)

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Tesseract
- Bounded by regular polyhedra
- All *cells* are congruent
Introduction

Polygons, Polyhedra, Polychora

Higher Dimension Polytopes

Polygons

Polyhedra

Polychora

Regular Convex Polyhedra (The Platonic Solids)

Square

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- All sides are congruent
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Classifying Regular Polytopes in Dimension 4 and Beyond
Regular Convex Polychora
Polytopes become increasingly difficult to visualize in higher dimensions.
Higher Dimension Polytopes

- Polytopes become increasingly difficult to visualize in higher dimensions.
- Regular convex polytopes of higher dimensions follow all of the same rules and patterns!
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Regular convex polytopes of higher dimensions follow all of the same rules and patterns!

- An $n$-polytope is bounded by $(n - 1)$-polytopes.
Higher Dimension Polytopes

- Polytopes become increasingly difficult to visualize in higher dimensions.
- Regular convex polytopes of higher dimensions follow all of the same rules and patterns!
  - An $n$-polytope is bounded by $(n-1)$-polytopes.
  - The facets and vertex figures must each be regular and congruent.
Polytopes become increasingly difficult to visualize in higher dimensions.

Regular convex polytopes of higher dimensions follow all of the same rules and patterns!

- An $n$-polytope is bounded by $(n - 1)$-polytopes.
- The facets and vertex figures must each be regular and congruent.
- The angles formed by the facets must be congruent.
Polytopes become increasingly difficult to visualize in higher dimensions.

Regular convex polytopes of higher dimensions follow all of the same rules and patterns!

- An $n$-polytope is bounded by $(n - 1)$-polytopes.
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- The angles formed by the facets must be congruent.
- The can be represented by a Schlafli symbol.
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- The angles formed by the facets must be congruent.
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There are only three regular convex polytopes in each dimension $n \geq 5$. 
Future Work

- Properties of $n$-polytopes ($n \geq 4$)
- Practical uses for this information
- Star polytopes
Thank you for coming!
References and Works Consulted


Various pages from Wikipedia.org were consulted