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Custom Locks: Counting the Combinations

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1. Introduction

Before we look at the focus of this presentation, it is necessary to understand the objects we will be dealing with. Instead of the traditional combination lock with clockwise and counterclockwise turns, it is best to visualize a keypad on, say, a door or a safe.
The rules for our combination lock are as follows:

1. All numbers must be used in a combination.
2. Each number can be used only once.
3. The order does matter, so we are technically counting permutations. However, since it is a "combination lock," we will be referring to the possible entries as combinations.

Counting the possible entries in a combination lock is trivial when we recognize that any lock with buttons 1 to n will have precisely $n!$ combinations. In this paper, we will explore the different methods of counting combinations when the buttons on our lock can be pushed simultaneously. Using recurrence relations and generating functions to derive formulas, we see that it is possible to count the number of combinations of any such lock with buttons 1 to n.

2. Combinations

Suppose there are three numbers (1, 2, and 3) in a typical combination lock. Then we have three choices for the first number, two choices for the second, and there will one left as the third number. We can show this combination as a sequence of disjoint, nonempty subsets.

$$(\{1\},\{2\},\{3\}) \quad , \quad (\{1\},\{3\},\{2\}) \quad , \quad (\{2\},\{1\},\{3\}),$$
$$(\{2\},\{3\},\{1\}) \quad , \quad (\{3\},\{1\},\{2\}) \quad , \quad (\{3\},\{2\},\{1\}).$$

Thus, we have that there are $3 * 2 * 1$ or $3! = 6$ possible combinations by the multiplication principle. It is reasonable then to conclude that if a combination lock has buttons 1 through n, there are $n!$ possible combinations.
Suppose we have the same lock as before (buttons 1, 2, and 3), but now the buttons can be pushed simultaneously.

$$(\{1\},\{2\},\{3\}) \quad , \quad (\{1\},\{3\},\{2\}) \quad , \quad (\{2\},\{1\},\{3\}),$$
$$(\{2\},\{3\},\{1\}) \quad , \quad (\{3\},\{1\},\{2\}) \quad , \quad (\{3\},\{2\},\{1\}),$$

$$(\{1,2\},\{3\}) \quad , \quad (\{1,3\},\{2\}) \quad , \quad (\{2,3\},\{1\}),$$

$$(\{1\},\{2,3\}) \quad , \quad (\{2\},\{1,3\}) \quad , \quad (\{3\},\{1,2\}) \quad , \quad (\{1,2,3\}).$$

Counting each collection, we can see that there are 13 possible combinations. For a general case, let a_n be the number of combinations for a lock with n buttons. If $n = 0$, then the only reasonable possibility is the empty sequence, thus $a_0 = 1$. Now, a possible combination will consist of a collection of k buttons that are pushed simultaneously (where $1 \leq k \leq n$) with $n - k$ buttons remaining to complete the combination. We can write this in recursive form as follows:

$$a_0 = 1, \;\; a_n = \binom{n}{1} a_{n-1} + \binom{n}{2} a_{n-2} + \cdots + \binom{n}{n} a_0 \;\text{ for } n > 0.$$

Thus, we can begin to find values for a_n. Notice that when we fill in the formula for the binomial coefficients, each term of the recurrence relation has a common factor of $n!$.

$$a_n = n! \left(\frac{a_{n-1}}{1!(n-1)!} + \frac{a_{n-2}}{2!(n-2)!} + \cdots + \frac{a_0}{n!0!} \right).$$

Now, let $b_n = \dfrac{a_n}{n!}$. This produces a new recurrence relation for b_n:

$$b_0 = 1, \quad b_n = b_{n-1} + \frac{b_{n-2}}{2!} + \cdots + \frac{b_0}{n!} \;\text{ for }\; n > 0.$$

Since the recurrence relation for b_n is slightly simpler than that of a_n, we can solve it to ultimately find a formula for a_n. The first step is to set some bounds for b_n.

In order to complete the proof, we will need the following definitions:

Definition 2.1 (Stewart, 2008) *(Taylor's Remainder Theorem) Suppose that f is $n+1$ times differentiable and let R_n denote the difference between $f(x)$ and the Taylor polynomial of degree n for $f(x)$ centered at a. Then*

$$R_n(x) = f(x) - T_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between x and a.

Definition 2.2 (Stewart, 2005) *(Sum of the Maclaurin Series for e^x)*

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ for all } x.$$

Theorem 2.3 (Velleman, Call, 1995) *For all n, $\frac{1}{2(\ln 2)^n} \leq b_n \leq \frac{1}{(\ln 2)^n}$.*

Proof: We will use strong induction on n.

Base Case: Let $n = 0$. Recall $b_0 = 1$.

$$\Rightarrow \frac{1}{2(\ln 2)^0} \leq b_0 \leq \frac{1}{(\ln 2)^0}$$

$$\Rightarrow \frac{1}{2} \leq 1 \leq 1$$

Thus, the theorem statement holds true at $n = 0$.

Induction Hypothesis: Assume the theorem statement holds true for all $n \geq 0$.

First, we start with the upper bound:

$$b_n = b_{n-1} + \frac{b_{n-2}}{2!} + \cdots + \frac{b_0}{n!}$$

$$\leq \frac{1}{(\ln 2)^{n-1}} + \frac{1}{2!(\ln 2)^{n-2}} + \cdots + \frac{1}{n!} \qquad \text{(By our Induction Hypothesis)}$$

$$= \frac{1}{(\ln 2)^n}\left(\ln 2 + \frac{(\ln 2)^2}{2!} + \cdots + \frac{(\ln 2)^n}{n!}\right) \text{ (Factoring } \frac{1}{(\ln 2)^n} \text{ out of each piece)}$$

$$\leq \frac{1}{(\ln 2)^n}\left(e^{\ln 2} - 1\right) \qquad \text{(Sum of the Maclaurin series for } e^x\text{)}$$

$$= \frac{1}{(\ln 2)^n}.$$

Then consider the lower bound:

$$b_n = b_{n-1} + \frac{b_{n-2}}{2!} + \cdots + \frac{b_0}{n!}$$

$$\geq \frac{1}{2(\ln 2)^{n-1}} + \frac{1}{2!2(\ln 2)^{n-2}} + \cdots + \frac{1}{(n-1)!2\ln 2} + \frac{1}{n!} \qquad \text{(By our Induction Hypoth-}$$

esis)
$= \dfrac{1}{2(\ln 2)^n}\left(\ln 2 + \dfrac{(\ln 2)^2}{2!} + \cdots + \dfrac{(\ln 2)^{n-1}}{(n-1)!} + \dfrac{2(\ln 2)^n}{n!}\right)$ (Factoring $\dfrac{1}{2(\ln 2)^n}$ out of each piece)

Using Taylor's Remainder Theorem, consider $e^{\ln 2} = 1 + \ln 2 + \dfrac{(\ln 2)^2}{2!} + \cdots + \dfrac{(\ln 2)^{n-1}}{(n-1)!} + \dfrac{e^c(\ln 2)^n}{n!}$, where $c \in \mathbb{R}$ and $0 < c < \ln 2$.
$\leq 1 + \ln 2 + \dfrac{(\ln 2)^2}{2!} + \cdots + \dfrac{(\ln 2)^{n-1}}{(n-1)!} + \dfrac{2(\ln 2)^n}{n!}$ (Since $e^c < e^{\ln 2} = 2$ by Taylor's Remainder Theorem)

Then $b_n \geq \dfrac{1}{2(\ln 2)^n}\left(\ln 2 + \dfrac{(\ln 2)^2}{2!} + \cdots + \dfrac{(\ln 2)^{n-1}}{(n-1)!} + \dfrac{2(\ln 2)^n}{n!}\right)$

$\geq \dfrac{1}{2(\ln 2)^n}\left(e^{\ln 2} - 1\right)$ (Sum of the Maclaurin series for e^x)

$= \dfrac{1}{2(\ln 2)^n}$

Thus, $\dfrac{1}{2(\ln 2)^n} \leq b_n \leq \dfrac{1}{(\ln 2)^n}$ for all $n > 0$.

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A reasonable way to look at the sequence (b_n) is to define a generating function as follows:

$$f(x) = \sum_{n=0}^{\infty}(b_n)x^n.$$

By Theorem 2.3, we can show that the sum converges absolutely for $|x| < \ln(2)$.

Proof:

By the ratio test, $f(x) = \sum_{n=0}^{\infty}(b_n)x^n$. converges when
$\limsup\left(\left|\dfrac{b_{n+1}x^{n+1}}{b_n x^n}\right|\right) = \limsup\left(\dfrac{b_{n+1}}{b_n}|x|\right) < 1.$
Case 1: $\dfrac{1/2(\ln 2)^{n+1}}{1/2(\ln 2)^n}|x| = \dfrac{1}{2(\ln 2)}|x| < 1 \Rightarrow |x| < 2(\ln 2).$

Case 2: $\dfrac{1/(\ln 2)^{n+1}}{1/(\ln 2)^n}|x| = \dfrac{1}{(\ln 2)}|x| < 1 \Rightarrow |x| < \ln 2.$

Therefore, by the ratio test and Theorem 2.3, $\sum_{n=0}^{\infty}(b_n)x^n$ converges absolutely for $|x| < \ln 2$.

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Now, by way of our recurrence relation for b_n, we can solve for $f(x)$.

$f(x) = \sum_{n=0}^{\infty}(b_n)x^n$

$\Rightarrow f(x) = b_0 + \sum_{n=1}^{\infty}(b_n)x^n$ (Pulling out the first term in the sum)

$= 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{b_{n-k}}{k!}x^n$ ($b_0 = 1$, Definition of b_n)

$= 1 + \sum_{k=1}^{\infty}\sum_{n=k}^{\infty}\frac{b_{n-k}}{k!}x^n$

Here, we note that even though the bounds have changed, the value remains the same as each sum ultimately includes all terms from 1 to infinity. Now, using the rules of exponents, we can separate our x term and then proceed to separate our sums as follows:

$= 1 + \left(\sum_{k=1}^{\infty}\frac{x^k}{k!}\right)\left(\sum_{n=k}^{\infty}b_{n-k}x^{n-k}\right)$ (Separating the sums)

$= 1 + \left(\sum_{k=1}^{\infty}\frac{x^k}{k!}\right)\left(\sum_{n=0}^{\infty}b_n x^n\right)$ (Changing the bounds of the second sum)

$= 1 + (e^x - 1)f(x)$ (Sum of the Maclaurin series for e^x, Definition of $f(x)$)

$\Rightarrow f(x) - (e^x - 1)f(x) = 1 \Rightarrow f(x)(1 - e^x + 1) = 1 \Rightarrow f(x)(2 - e^x) = 1 \Rightarrow f(x) = \dfrac{1}{2 - e^x}$.

Thus, $f(x) = \dfrac{1}{2 - e^x}$ when $|x| < \ln 2$.

Since b_n is the coefficient of x^n in the Maclaurin series for $f(x)$, we have

$$b_n = \frac{f^{(n)}(0)}{n!}$$

Remember that $a_n = n! * b_n$, so we have now proven the following theorem:

Theorem 2.4 (Velleman, Call, 1995) *For all n,*

$$a_n = \frac{d^n}{dx^n}\left(\frac{1}{2-e^x}\right)\bigg|_{x=0}$$

Thus, to find values for a_n, we must find derivatives of our function $\dfrac{1}{2-e^x}$. To make this easier, we can write our function as a geometric series:

$$\frac{1}{2-e^x} = \frac{\frac{1}{2}}{1-\frac{e^x}{2}} = \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{e^x}{2}\right)^k, |x| < \ln 2.$$

If we differentiate term by term, we find that

$$\frac{d^n}{dx^n}\left(\frac{1}{2-e^x}\right) = \frac{1}{2}\sum_{k=0}^{\infty}k^n\left(\frac{e^x}{2}\right)^k, |x| < ln2.$$

By Theorem 2.4, we can state a new formula for a_n

Theorem 2.5 (Velleman, Call, 1995) *For all n,*

$$a_n = \frac{1}{2}\sum_{k=0}^{\infty}\frac{k^n}{2^k}.$$

If we consider our sum in terms of area, each term can be though of as blocks of various heights. Thus, it is reasonable to use the area underneathe a curve to approximate the value of our sum. Hence, we can estimate a_n with an improper integral using the same bounds.

$$\int_0^{\infty}\frac{x^n}{2^x}dx$$

First, to avoid integrating by parts, we need to define a new function.

Definition 2.6 *(The Gamma Function $\Gamma(x)$)*

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt = \int_0^{\infty}\frac{t^{x-1}}{e^t}dt = (n-1)!$$

Now, we can evaluate the integral using the gamma function.

$\Gamma(x) = \displaystyle\int_0^{\infty} t^{x-1}e^{-t}dt = \int_0^{\infty}\frac{t^{x-1}}{e^t}dt$ (Definition of $\Gamma(x)$)

$\Gamma(x+1) = \int_0^\infty \frac{t^{(x+1)-1}}{e^t} dt = \int_0^\infty \frac{t^x}{e^t} dt$ (Definition of $\Gamma(x+1)$)

Thus, $\Gamma(n+1) = \int_0^\infty \frac{u^n}{e^u} du = n!$

$\int_0^\infty \frac{u^n}{e^u} du = \int_0^\infty \frac{(x \ln 2)^n}{e^{x \ln 2}} dx (\ln 2)$ (Substituting $u = x \ln 2$)

$\Rightarrow \frac{1}{(\ln 2)^{n+1}} \int_0^\infty \frac{x^n (\ln 2)^{n+1}}{e^{\ln 2^x}} dx$

$= \int_0^\infty \frac{x^n (\ln 2)^{n+1}}{(\ln 2)^{n+1} 2^x} dx$

$= \int_0^\infty \frac{x^n}{2^x} dx$

Thus, $\int_0^\infty \frac{x^n}{2^x} dx = \frac{1}{(\ln 2)^{n+1}} \int_0^\infty \frac{u^n}{e^u} du = \frac{\Gamma(n+1)}{(\ln 2)^{n+1}} = \frac{n!}{(\ln 2)^{n+1}}$

Hence, by Theorem 2.5, $a_n = \frac{1}{2} \sum_{k=0}^{\infty} \frac{k^n}{2^k} \approx \frac{1}{2} \int_0^\infty \frac{x^n}{2^x} = \frac{n!}{2(\ln 2)^{n+1}}$

For reasonably small values of n, this approximation is extremely accurate.

$$a_9 = 7087261$$

$$\frac{9!}{2(\ln 2)^{10}} \approx 7087261.0016$$

That's a difference of only 0.0016. However, the error of approximation seems to fluctuate for different values of n. Thus, it becomes necessary to place bounds on the error.

Theorem 2.7 (Velleman, Call, 1995) *For all n,*

$$\frac{n!}{2(\ln 2)^{n+1}} - \frac{1}{2} \left(\frac{n}{e \ln 2} \right)^n < a_n < \frac{n!}{2(\ln 2)^{n+1}} + \frac{1}{2} \left(\frac{n}{e \ln 2} \right)^n.$$

Proof: Consider our function $g(x) = \frac{x^n}{2^x}$. First, we need to identify when our function is increasing and when it is decreasing by looking at its derivative.

$g(x) = \frac{x^n}{2^x}$

$g'(x) = \frac{n x^{n-1} 2^x - x^n 2^x \ln 2}{2^{2x}}$

Note that $\frac{dy}{dx}2^x = 2^x \ln 2$ by implicit differentiation.

Now, since we are only looking at positive x values, we know that g will be increasing when the left side of the numerator of g' is greater than the right side, and g will have a slope of 0 when the two sides are equal. Suppose the left side is greater.

$$nx^{n-1}2^x > x^n 2^x \ln 2$$
$$\Rightarrow nx^{n-1} > x^n \ln 2$$
$$\Rightarrow \frac{n}{x} > \ln 2$$
$$\Rightarrow \frac{n}{\ln 2} > x$$

Thus, g is increasing on $\left[0, \frac{n}{\ln 2}\right]$ and decreasing on $\left[\frac{n}{\ln 2}, \infty\right)$. Based on this evaluation, our function g will have a maximum value at $\frac{n}{\ln 2}$.

Now, consider $g\left(\frac{n}{\ln 2}\right) = \frac{\left(\frac{n}{\ln 2}\right)^n}{2^{\left(\frac{n}{\ln 2}\right)}}$

$= \left(\frac{n}{\ln 2}\right)^n \left(\frac{1}{2}\right)^{\frac{n}{\ln 2}}$

$= \left(\frac{n}{(\ln 2)2^{\frac{1}{ln 2}}}\right)^n$

Note that $2^{\frac{1}{\ln 2}} = e^{ln(2^{\frac{1}{\ln 2}})} = e^{\frac{1}{\ln 2}\ln 2} = e$.

Thus, $\left(\frac{n}{e \ln 2}\right)^n$ is the maximum error value on the interval $[0, \infty)$.

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For our next proof, we will need a new definition:

Definition 2.8 *(Stirling's Formula)*

$$\lim_{n\to\infty} \frac{\sqrt{2\pi n}(\frac{n}{e})^n}{n!} = 1$$

Using the bounds of the error of our approximation and Stirling's Formula, we can prove the following:

Corollary 2.9 (Velleman, Call, 1995) *(Accuracy of the Approximation)*

$$\lim_{n\to\infty} \frac{a_n}{n!/2(\ln 2)^{n+1}} = 1$$

Proof. By Theorem 2.7,

$$\frac{n!}{2(\ln 2)^{n+1}} - \frac{1}{2}\left(\frac{n}{e\ln 2}\right)^n < a_n < \frac{n!}{2(\ln 2)^{n+1}} + \frac{1}{2}\left(\frac{n}{e\ln 2}\right)^n$$

$$\Rightarrow \frac{n!}{2(\ln 2)^{n+1}} - \frac{\left(\frac{n}{e}\right)^n}{2(\ln 2)^n} < a_n < \frac{n!}{2(\ln 2)^{n+1}} + \frac{\left(\frac{n}{e}\right)^n}{2(\ln 2)^n} \quad \text{(Applying the exponent)}$$

$$\Rightarrow \frac{n!}{2(\ln 2)^{n+1}} - \frac{n!\left(\frac{n}{e}\right)^n \ln 2}{n!2(\ln 2)^{n+1}} < a_n < \frac{n!}{2(\ln 2)^{n+1}} + \frac{n!\left(\frac{n}{e}\right)^n \ln 2}{n!2(\ln 2)^{n+1}} \quad \text{(Multiplying by } \tfrac{n!}{n!}\text{)}$$

$$\Rightarrow -\frac{\ln 2(\frac{n}{e})^n}{n!} + 1 < \frac{a_n}{n!/2(\ln 2)^{n+1}} < \frac{\ln 2(\frac{n}{e})^n}{n!} + 1 \text{ (Factoring, division)}$$

$$\left|\frac{a_n}{n!/2(\ln 2)^{n+1}} - 1\right| < \frac{\ln 2(\frac{n}{e})^n}{n!} \text{ (Definition of absolute value)}$$

Now, by Stirling's Formula,

$$\lim_{n\to\infty} \frac{\ln 2(\frac{n}{e})^n}{n!} = \lim_{n\to\infty} \frac{\sqrt{2\pi n}}{\sqrt{2\pi n}} \frac{\ln 2(\frac{n}{e})^n}{n!} \text{ (Multiplying by } \tfrac{\sqrt{2\pi n}}{\sqrt{2\pi n}}\text{)}$$

$$= \lim_{n\to\infty} \frac{\ln 2}{\sqrt{2\pi n}} \frac{\sqrt{2\pi n}(\frac{n}{e})^n}{n!}$$

$$= \lim_{n\to\infty} \frac{\ln 2}{\sqrt{2\pi n}} \lim_{n\to\infty} \frac{\sqrt{2\pi n}(\frac{n}{e})^n}{n!}$$

$$= 0 * 1 = 0$$

❑

Thus, we see that the exact value a_n and our approximation are equal as n approaches infinity.

3. Permutations

In addition to estimating the sum in Theorem 2.5 to get approximations for a_n, we can evaluate the sum exactly. For $n \geq 0$, let

$$h_n(x) = \sum_{k=0}^{\infty} k^n x^k$$

By Theorem 3, $a_n = \dfrac{1}{2}h_n(\frac{1}{2})$. Now we can find a formula for a_n by using different n values for $h_n(x)$.

$$h_0(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \ -1 < x < 1.$$

Differentiating term by term, we see that

$h'_n(x) = \displaystyle\sum_{k=0}^{\infty} k^{n+1}x^{k-1}$ (By the power rule)

$\Rightarrow xh'_n(x) = \displaystyle\sum_{k=0}^{\infty} k^{n+1}x^k = h_{n+1}(x)$ (Definition of $h_n(x)$)

Now we can apply this recurrence repeatedly. Here are some formulas for $h_n(x)$ when $n \leq 5$:

$$h_0(x) = \frac{1}{1-x}$$
$$h_1(x) = \frac{x}{(1-x)^2}$$
$$h_2(x) = \frac{x+x^2}{(1-x)^3}$$
$$h_3(x) = \frac{x+4x^2+x^3}{(1-x)^4}$$
$$h_4(x) = \frac{x+11x^2+11x^3+x^4}{(1-x)^5}$$

Thus, $h_n(x)$ is always an n^{th} degree polynomial divided by $(1-x)^{n+1}$. We'll look at this pattern further by using a new notation. Let $A_{n,k}$ be the coefficient of x^k in the numerator of $h_n(x)$. For example, looking back at the recurrence, $A_{4,3} = 11$. Therefore, it makes sense to use the recurrence for $h_n(x)$ to find a recurrence for $A_{n,k}$.

Theorem 3.1 (Velleman, Call, 1995) *For all* $n \geq 1$,

$$h_n(x) = \frac{\displaystyle\sum_{k=1}^{n} A_{n,k}x^k}{(1-x)^{n+1}},$$

where $A_{n,k}$ *is given by the following recurrence relation:*

$$A_{n,1} = A_{n,n} = 1, \ A_{n+1,k} = kA_{n,k} + (n+2-k)A_{n,k-1} \ \text{ for } 2 \leq k \leq n.$$

Proof. We will proceed by way of induction.

Base Case: For $n = 1, A_{1,1} = 1, h_n(x) = \dfrac{x}{(1-x)^2}$.

Induction Hypothesis: Assume all of the conditions of the theorem.

Induction Step:

$$h_{n+1} = xh_n'(x) = x\frac{d}{dx}\left(\frac{\sum_{k=1}^{n} A_{n,k}x^k}{(1-x)^{n+1}}\right)$$

$$= x\frac{(1-x)^{n+1}\sum_{k=1}^{n} kA_{n,k}x^{k-1} + (n+1)(1-x)^n\sum_{k=1}^{n} A_{n,k}x^k}{(1-x)^{2n+2}} \quad \text{(By the Product/Chain Rule)}$$

$$= \frac{\sum_{k=1}^{n} kA_{n,k}(1-x)x^k + \sum_{k=1}^{n}(n+1)A_{n,k}x^{k+1}}{(1-x)^{n+2}} \quad \text{(Factoring } (1-x)^n \text{ and distributing } x\text{)}$$

Now, if we distribute the $(1-x)$ term in the numerator, we will have three separate sums. By factoring out $A_{n,k}x^{k+1}$ from two of the sums, we arrive at the following:

$$\frac{\sum_{k=1}^{n} kA_{n,k}x^k + \sum_{k=1}^{n}(n+1-k)A_{n,k}x^{k+1}}{(1-x)^{n+2}}.$$

Next, we will change the bounds of our sum. For the left piece, we evaluate $kA_{n,k}x^k$ at $k = 1$, which is just x, making our sum extend from $k = 2$ to infinity. For the right piece, we use the bounds $k = 2$ to infinity, but this time we choose to change each k in the sum to $k-1$. However, this means we need to evaluate $A_{n,k-1}x^k(n+2-k)$ at $k = n+1$, leaving us with x^{n+1}. Thus, we have

$$\frac{x + \sum_{k=2}^{n}(kA_{n,k} + (n+2-k)A_{n,k-1})x^k + x^{n+1}}{(1-x)^{n+2}}.$$

Finally, by our induction hypothesis and multiplying our x and x^{n+1} terms back into the sum, we reach our expected outcome:

$$\frac{\sum_{k=1}^{n+1} A_{n+1,k}x^k}{(1-x)^{n+2}}.$$

❏

The recurrence relation given in Theorem 3.1 allows us to compute coefficients $A_{n,k}$ for $1 \le k \le n$ if we know the coefficients $A_{n-1,k}$ for $1 \le k \le n-1$. Now we can arrange the numbers in a triangular table (similar in format to Pascal's Triangle) with an n^{th} row of

$$A_{n,1},\ A_{n,2},\ ...,\ A_{n,n}$$

. By starting with $A_{1,1} = 1$, we can use Theorem 3.1 to produce each row that follows. The first five rows are given below.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 4 & & 1 & & \\ & 1 & & 11 & & 11 & & 1 & \\ 1 & & 26 & & 66 & & 26 & & 1 \end{array}$$

The triangle is symmetric, so perhaps $A_{n,k} = A_{n,n+1-k} \quad \forall\ k \ni 1 \le k \le n$. Further, if we add the rows, (similar to Pascal's Triangle, where the n^{th} row adds up to 2^n), we see that the sums are 1, 2, 6, 24, and 120. Note that these are the numeric values of 1!, 2!, 3!, 4!, and 5!. This suggests that we should look for an interpretation of $A_{n,k}$ ($\forall\ k \ni 1 \le k \le n$) in terms of a partition of the set of permutations $\{1, 2, ..., n\}$. Now, we will let the term n-permutation denote a permutation of $\{1, 2, ..., n\}$. Consider a permutation $s_1 s_2 ... s_n$ of this form. We proceed by counting the number of increasing runs in the sequence from left to right. For example, the permutation 132564 has three increasing runs: 13, 256, and 4.

Proposition 3.2 (Comtet, 1974) *For all n and k such that $1 \le k \le n$, the number of n-permutations with k increasing runs is $A_{n,k}$.*

The numbers $A_{n,k}$ and the interpretation used for them in Proposition 3.2 are called the Eulerian numbers. Thus, we can refer to our triangular arrangement of the numbers $A_{n,k}$ as Euler's Triangle. Hence, our conjectures on the triangle can be verified.

Corollary 3.3 (Comtet, 1974) *For all $n \ge 1$,*

(a) $\displaystyle\sum_{k=1}^{n} A_{n,k} = n!$ *and*

(b) $A_{n,k} = A_{n,n+1-k}\ \forall\ k \ni 1 \le k \le n$

Proof. For part (a), we know this is true by Proposition 3.2 since the sum of a row in Euler's triangle yields the exact same result. As for part (b), this is a known attribute of Eulerian numbers. ❑

Now, we can look at the original lock problem again. By Theorem 2.5 and Theorem 3.1, the number of combinations for a lock with $n \geq 1$ satisfies the following equation:

$$a_n = \frac{1}{2}h_n(\tfrac{1}{2}) = \frac{1}{2}\sum_{k=1}^{n} A_{n,k}\left(\frac{1}{2}\right)^{k-n-1} = \sum_{k=1}^{n} A_{n,k}2^{n-k} \quad (1)$$

By the symmetry of Euler's Triangle, we also have that

$$a_n = \sum_{k=1}^{n} A_{n,n+1-k}2^{(n+1-k)-1} = \sum_{k=1}^{n} A_{n,k}2^{k-1} \quad (2)$$

Given a lock combination, we can write a corresponding n-permutation. We record which numbers are pressed (in the order that they are pressed) and list the buttons being pushed simultaneously as an increasing run.

Consider the combination $(\{2,4\},\{5\},\{1,3\})$. This would correspond to the permutation 24513.

A lock combination with l steps will produce a permutation that can have at most l increasing runs. Now, if we do this for every lock combination, then each permutation with precisely k increasing runs will appear 2^{n-k} times.

Proof. Let σ be an n-permutation with k increasing runs. We will separate this permutation into segments that represent the steps in the corresponding combination. Since we have k increasing runs, we would need at least $k-1$ lines to separate them all. Now consider the numbers themselves. To separate them, it would take $n-1$ lines to separate them all (of which, $k-1$ of them are already filled). Thus, there are $(n-1)-(k-1) = n-k$ spaces left open between each of the numbers in the permutation. Finally, we can choose any subset of these lines to form our lock combination corresponding to σ. ❑

For example, the permutation 24153 that we used ealier has two increasing runs (245 and 13). Thus, this permutation corresponds to $2^{5-2} = 8$ distinct lock combinations as shown below:

$$(\{2,4,5\},\{1,3\}),\ (\{2\},\{4,5\},\{1,3\}),\ (\{2,4\},\{5\},\{1,3\}),\ (\{2\},\{4\},\{5\},\{1,3\})$$

$$(\{2,4,5\},\{1\},\{3\}),\ (\{2\},\{4,5\},\{1\},\{3\})$$

$$(\{2,4\},\{5\},\{1\},\{3\}),\ (\{2\},\{4\},\{5\},\{1\},\{3\})$$

Next, we will implement another common formula for the Eulerian numbers that uses binomial coefficients.

Theorem 3.4 (Comtet, 1974) *For all* $k \ni 1 \leq k \leq n$*,*

$$A_{n,k} = \sum_{i=0}^{k-1} (-1)^i \binom{n+1}{i} (k-i)^n$$

Using Theorem 3.4, Equation 1, and Equation 2, we see that there are two more formulas for a_n.

Theorem 3.5 (Velleman, Call, 1995) *For all* $n \geq 1$*,*

$$a_n = \sum_{k=1}^{n} \sum_{i=0}^{k-1} (-1)^i \binom{n+1}{i} (k-i)^n 2^{n-k} = \sum_{k=1}^{n} \sum_{i=0}^{k-1} (-1)^i \binom{n+1}{i} (k-i)^n 2^{k-1}.$$

The last solution we have for our combination lock problem involves another common group of numbers.

Definition 3.6. *(Stirling Numbers of the Second Kind)*
The number of unordered partitions of the set $\{1, 2, ..., n\}$ *into* k *nonempty subsets are called Stirling numbers of the second kind. They are denoted* $S(n, k)$*, where*

$$S(n,k) = \frac{1}{k!} \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} (k-1)^n, \ \forall \ k \ni 1 \leq k \leq n.$$

Thus, the number of lock combinations with k steps can be denoted as $k!S(n, k)$. Summing over all of k logically leads to another formula.

Theorem 3.7 (Comtet, 1974) *For all* $n \geq 1$*,*

$$a_n = \sum_{k=1}^{n} k!S(n,k).$$

Additionally, the Stirling numbers of the second kind can be written in terms of binomial coefficients as follows:

Theorem 3.8 (Comtet, 1974) *For all* $1 \leq k \leq n$*,*

$$S(n,k) = \frac{1}{k!} \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} (k-i)^n.$$

By substituting the formula for $S(n,k)$ from Theorem 3.8 into Theorem 3.7, we obtain our last formula.

Theorem 3.9 (Velleman, Call, 1974) *For all* $n \geq 1$,

$$a_n = \sum_{k=1}^{n} \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} (k-i)^n.$$

4. Conclusion

Hence, we have now shown that given a combination lock with buttons 1 to n, it is possible to compute the total number of possible combinations.

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