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Playing with Permutations: Examining Mathematics in Children's Toys

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By

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1. Introduction

In John P. Bonomo's and Carolyn K. Cuff's paper *How Do You Stack Up?* a mathematical problem was posed. This question was regarding a common children's toy known as a stacking ring tower. The problem the authors addressed came as a result of a common occurrence: the event that not all of the rings are placed on the tower in the "proper" order. When the rings are placed on the tower in a variation of that "proper" order, some of them will inevitably stick over the top of the tower. The problem the authors decided to tackle was to find the average number of rings that stick over the top of the tower when examining all possible placements of the rings. The authors found a solution and proved their solution to be true within their paper.

At the beginning of this project, it was my goal to solve the same problem independently of Bonomo and Cuff, and then to compare my results with theirs. I ended up taking a very different approach, but in the end, my work ended up corresponding with theirs. In my paper, I will explain my thought process and my methods for solving this problem. I will guide the reader through my strategies and explain how they did (or did not) work out for me. We will begin with some basic definitions and explanation of the problem in greater detail, and then commence with my research.

2. Background Information

Definition 2.1 (Stacking Ring Tower) *A stacking ring tower is a common children's toy, which consists of a tapered rod and a certain number of rings. The rod is larger at the bottom than it is at the top. We will use n to specify the height of the rod, and consequently, the number of rings. Each ring has exactly one correct position on the rod where it fits snugly, and when all rings are placed on the rod so that they lay in their respective correct positions, the height of the stacked rings is the same as the height of the tower. The largest ring will be referred to as Ring 1, the second largest ring will be referred to as Ring 2, and so on until all of the rings have been named.*

We note that when Ring i is dropped on the tower, there are three possible results:

- the ring will fall to its proper position (position i);
- the ring will fall to a position higher on the rod than its proper position; or
- the ring will fall to a position higher than the rod itself.

The first result will occur when all the rings dropped before Ring i have come to rest below position i. The second result will occur when any of the rings dropped before Ring i have come to rest at or above position i, but below position n. The third result will occur when any of the rings dropped before Ring i have come to rest at or above position n.

Definition 2.2 (Extent) *The extent of a particular permutation of the n rings on the stacking ring tower describes the number of rings which lay above the topmost position on the rod (position n) after all n rings have been dropped. Note that in order for an extent to be greater than 0, the nth position of the tower must be filled and we must have more rings to place on the tower. The length or size of an extent may be denoted by the variable e.*

Definition 2.3 (Average Extent) *The average extent of all permutations of n rings refers to the sum of the extents produced by each permutation of the n rings divided by $n!$, the total number of permutations of the n rings.*

Remark 2.4 *To describe a particular permutation, I will denote the order of the rings being dropped with the number of each ring dropped separated by a dash. For example, if I want to describe the scenario where I have four rings to drop and I drop Ring 2 first, followed by Ring 3, then Ring 1, and I finish by dropping Ring 4, I would denote this particular permutation as* 2-3-1-4.

3. The Primary Problem

The primary question posed by Bonomo and Cuff was *What is the average extent of all permutations of n rings?* In my attempt to solve this problem, my first instinct was to manually calculate the average extent of all permutations of n rings for small values of n and then to search for patterns with the hope of predicting a formula for the general case of n rings. Using this method, I was able to find and explain various patterns in the extents and why they occur. The following lemmas and corollaries are what I found.

Lemma 3.1 *If a stacking ring tower is of height n, then the possible values of the extents of permutations of those n rings range from 0 to $n-1$.*

Proof: Suppose a stacking ring tower has height n. Suppose I want to produce a permutation that will have extent e where $0 \leq e \leq n-1$. To do this, I can drop Ring $e+1$ first, Ring $e+2$ second, Ring $e+3$ third, and then continue in this manner until I have dropped Ring n. At this point, position n will be filled with Ring n, and I will have dropped all but e rings. This means that regardless of the order in which I drop those e rings, I will have an extent of e.

Since I must drop all n rings and there are only n positions on the tower, I cannot have an extent less than 0. Also, since in order to build an extent I must first fill position

n, I must use at least one ring before the extent starts to grow. Since I only have n rings at my disposal, the largest possible extent is $n-1$. By this and the argument above, the possible values of extents of permutations of n rings ranges from 0 to $n-1$.

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Remark 3.2 *It is important to note that the only way to get an extent of* 0 *is to place the rings onto the tower in the "proper" order. As there is only one way to do this, there is exactly one permutation which results in an extent of* 0*.*

Lemma 3.3 *If a tower has height n, the extent of a permutation will be $n-1$ if and only if the permutation begins with Ring n.*

Proof: Suppose a tower has n rings.

($\Rightarrow$) Suppose a permutation has extent $n-1$. By the definition of extent and the fact that we have n rings to drop, we know there is exactly one ring on the tower. By the definition of extent, we know this ring must fill position n in order to have the remaining $n-1$ rings make up the extent. Thus the only possible ring we could have on the tower is Ring n, meaning we must have dropped Ring n first.

($\Leftarrow$) Suppose a permutation of the n rings begins with Ring n, meaning we drop Ring n first. By the definition of extent, the remaining rings will make up the extent. Since we have only dropped one of the n rings at our disposal, we can conclude the extent will be $n-1$.

Thus if a tower has height n, the extent of a permutation will be $n-1$ if and only if the permutation begins with Ring n.

❏

Corollary 3.4 *If a tower has height n, there will be exactly $(n-1)!$ permutations with extent $n-1$.*

Proof: Suppose a tower has height n. By Lemma 3.3, the only permutations with an extent of $n-1$ are those which begin with Ring n. When we drop Ring n first, there are $n-1$ rings left to rearrange and thus there are exactly $(n-1)!$ permutations with extent $n-1$.

❏

Lemma 3.5 *If the $(n-1)$th ring on a tower with height n is dropped first, then the extent will be of size $n-2$.*

Proof: Suppose a tower has height n and suppose we drop Ring $n-1$ on the tower first. That ring will slide to position $n-1$ and stop, leaving position n above it empty. Since all remaining rings left to be dropped will be forced to stop at position n and they all will fit on position n, any of the remaining rings can fill position n. By the definition of extent, we know that any rings dropped after these first two rings will form the extent. Since there will be $n-2$ of these rings, we may conclude that any permutation beginning with Ring $n-1$ will form an extent of $n-2$, as desired.

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It is tempting to think that like in Lemma 3.3 and its corollary, there will be exactly $(n-2)!$ permutations with extent $n-2$. However, a quick counterexample shows that this is not the case. What we want to show is that there is a permutation beginning with a ring other than Ring $n-1$ that will result in an extent of $n-2$. Suppose $n=5$. Consider the permutation 3-5-1-2-4. This permutation begins with Ring 3, or Ring $n-2$ in this case, and will form an extent of 3, or $n-2$. Thus there is a permutation beginning with a ring other than Ring $n-1$ that forms an extent of $n-2$. Thus we cannot conclude that dropping Ring $n-1$ first will provide all permutations with extent $n-2$, meaning there will not be exactly $(n-2)!$ permutations with extent $n-2$.

From the previous two lemmas, it might seem that a pattern is forming and that we might be able to conclude that if Ring $n-2$ is dropped first that we will always get an extent of $n-3$. However, this is not the case. As a counterexample, suppose $n=5$ and we look at the permutations 3-1-2-4-5 and 3-5-1-2-4. The first permutation forms an extent of 2, or $n-3$ in this case, and the second permutation forms an extent of 3, or $n-2$. This shows that dropping ring $n-2$ first does not guarantee an extent of $n-3$.

Lemma 3.6 *The set of all permutations on a tower with n rings has a 1-1 correspondence to the set of all permutations in which Ring 1 is dropped first on a tower with $n+1$ rings.*

Proof: Suppose a tower has a height of $n+1$. Consider the permutations where Ring 1 is dropped first. When this ring is dropped, it falls to position 1 at the very bottom of the tower. This results in n remaining empty positions above Ring 1, with no empty positions below that first ring. It is clear then that the permutations of the remaining n rings will now correspond exactly to all permutations of a tower with n rings. We can thus conclude that the set of all permutations of a tower with n rings has a 1-1 correspondence to the set of all permutations in which Ring 1 is dropped first on a tower with $n+1$ rings.

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While these patterns were interesting and certainly provided some information about the extents of certain permutations, the patterns I was finding were only telling me about the permutations where I dropped rings $1, n-1$, and n first which will essentially only be helpful when n is small. And so, unfortunately, this method of exploration did not allow me to find a closed formula to describe the average extent for a rod with n rings.

4. Examining Extents

At this point, I decided to explore the problem with a different goal. This time I focused on finding ways to count the number of permutations which would give me a certain extent for various cases of n. I used my previous work to group the permutations by the size of each of their extents. The following demonstrates this for the cases of $n = 1, n = 2, n = 3, n = 4$, and $n = 5$.

Lemma 4.1 *The average extent of a stacking ring tower when $n = 1$ is 0.*

Proof: When $n = 1$, the only possible extent is 0 by Lemma 3.1. There is exactly one way to get this extent by Remark 3.2, so this case is trivial. We can easily see that the average extent in this case is 0. ❑

Lemma 4.2 *The average extent of a stacking ring tower when $n = 2$ is 0.5*

Proof: When $n = 2$, the only possible extents are 0 and 1. We get the extent of 0 when ring 1 is dropped first (meaning the rings are dropped in order). We get the extent of 1 when ring 2 is dropped first. Thus the number of permutations with an extent between 0 and $n - 1 = 1$ is 2. Since $2 = 2!$ we have accounted for all permutations. From here, we can calculate that the average extent is $\frac{0+1}{2} = \frac{1}{2} = 0.5$. ❑

Lemma 4.3 *The average extent of a stacking ring tower when $n = 3$ is $\frac{7}{6}$.*

Proof: When $n = 3$, we have possible extents of 0, 1, and 2 by Lemma 3.1. The following is a table showing the permutations which correspond to specific extents in the $n = 3$ case.

	Extent=0	Extent=1	Extent=2
Permutation	1-2-3	1-3-2	3-1-2
Permutation	-	2-1-3	3-2-1
Permutation	-	2-3-1	-
Number of Permutations	1	3	2

When we add the values in the Number of Permutations row, we get $1 + 3 + 2 = 6$ and since $6 = 3!$ we have accounted for all permutations. We then calculate that the average extent is $\frac{1 \cdot 0 + 3 \cdot 1 + 2 \cdot 2}{6} = \frac{7}{6} \approx 1.167$ ❑

Lemma 4.4 *The average extent of a stacking ring tower when $n = 4$ is $\frac{45}{24}$.*

Proof: We note that when $n = 4$ the possible extents are 0, 1, 2, and 3 by Lemma 3.1. The following is a table showing the permutations which correspond to specific extents in the $n = 4$ case.

	Extent=0	Extent=1	Extent=2	Extent=3
Permutation	1-2-3-4	1-2-4-3	1-4-2-3	4-1-2-3
Permutation	-	1-3-2-4	1-4-3-2	4-1-3-2
Permutation	-	1-3-4-2	2-4-1-3	4-2-1-3
Permutation	-	2-1-3-4	2-4-3-1	4-2-3-1
Permutation	-	2-1-4-3	3-1-2-4	4-3-1-2
Permutation	-	2-3-1-4	3-1-4-2	4-3-2-1
Permutation	-	2-3-4-1	3-2-1-4	-
Permutation	-	-	3-2-4-1	-
Permutation	-	-	3-4-1-2	-
Permutation	-	-	3-4-2-1	-
Number of Permutations	1	7	10	6

Sure enough the sum of the values in the Number of Permutations row is $24 = 6!$ and so we have accounted for all permutations. From here we calculate the average extent to be $\frac{1 \cdot 0 + 7 \cdot 1 + 10 \cdot 2 + 6 \cdot 3}{24} = \frac{45}{24} = 1.875$ ❏

Lemma 4.5 *The average extent of a stacking ring tower when $n = 5$ is $\frac{313}{120}$.*

Proof: When $n = 5$, the possible extents are 0, 1, 2, 3, and 4 by Lemma 3.1.

	Extent=0	Extent=1	Extent=2	Extent=3	Extent=4
Permutation	1-2-3-4-5	1-2-3-5-4	1-2-5-3-4	1-5-2-3-4	5-1-2-3-4
Permutation	-	1-2-4-3-5	1-2-5-4-3	1-5-2-4-3	5-1-2-4-3
Permutation	-	1-2-4-5-3	1-3-5-2-4	1-5-3-2-4	5-1-3-2-4
Permutation	-	1-3-2-4-5	1-3-5-4-2	1-5-3-4-2	5-1-3-4-2
Permutation	-	1-3-2-5-4	1-4-2-3-5	1-5-4-2-3	5-1-4-2-3
Permutation	-	1-3-4-2-5	1-4-2-5-3	1-5-4-3-2	5-1-4-3-2
Permutation	-	1-3-4-5-2	1-4-3-2-5	2-5-1-3-4	5-2-1-3-4
Permutation	-	2-1-3-4-5	1-4-3-5-2	2-5-1-4-3	5-2-1-4-3
Permutation	-	2-1-3-5-4	1-4-5-2-3	2-5-3-1-4	5-2-3-1-4
Permutation	-	2-1-4-3-5	1-4-5-3-2	2-5-3-4-1	5-2-3-4-1
Permutation	-	2-1-4-5-3	2-1-5-3-4	2-5-4-1-3	5-2-4-1-3
Permutation	-	2-3-1-4-5	2-1-5-4-3	2-5-4-3-1	5-2-4-3-1
Permutation	-	2-3-1-5-4	2-3-5-1-4	3-5-1-2-4	5-3-1-2-4
Permutation	-	2-3-4-1-5	2-3-5-4-1	3-5-1-4-2	5-3-1-4-2
Permutation	-	2-3-4-5-1	2-4-1-3-5	3-5-2-1-4	5-3-2-1-4
Permutation	-	-	2-4-1-5-3	3-5-2-4-1	5-3-1-4-2
Permutation	-	-	2-4-3-1-5	3-5-4-1-2	5-3-4-1-2
Permutation	-	-	2-4-3-5-1	3-5-4-2-1	5-3-4-2-1
Permutation	-	-	2-4-5-1-3	4-1-2-3-5	5-1-2-3-4
Permutation	-	-	2-4-5-3-1	4-1-2-5-3	5-1-2-4-3
Permutation	-	-	3-1-2-4-5	4-1-3-2-5	5-1-3-2-4
Permutation	-	-	3-1-2-5-4	4-1-3-5-2	5-1-3-4-2
Permutation	-	-	3-1-4-2-5	4-1-5-2-3	5-1-4-2-3
Permutation	-	-	3-1-4-5-2	4-1-5-3-2	5-1-4-3-2
Permutation	-	-	3-1-5-2-4	4-2-1-3-5	-
Permutation	-	-	3-1-5-4-2	4-2-1-5-3	-
Permutation	-	-	3-2-1-4-5	4-2-3-1-5	-
Permutation	-	-	3-2-1-5-4	4-2-3-5-1	-
Permutation	-	-	3-2-4-1-5	4-2-5-1-3	-
Permutation	-	-	3-2-4-5-1	4-2-5-3-1	-
Permutation	-	-	3-2-5-1-4	4-3-1-2-5	-
Permutation	-	-	3-2-5-4-1	4-3-1-5-2	-
Permutation	-	-	3-4-1-2-5	4-3-2-1-5	-
Permutation	-	-	3-4-1-5-2	4-3-2-5-1	-
Permutation	-	-	3-4-2-1-5	4-3-5-1-2	-
Permutation	-	-	3-4-2-5-1	4-3-5-2-1	-
Permutation	-	-	3-4-5-1-2	4-5-1-2-3	-
Permutation	-	-	3-4-5-2-1	4-5-1-5-2	-
Permutation	-	-	-	4-5-2-1-3	-
Permutation	-	-	-	4-5-2-3-1	-
Permutation	-	-	-	4-5-3-1-2	-
Permutation	-	-	-	4-5-3-2-1	-
Number of Permutations	1	15	38	42	24

We see here that the sum of the values in the Number of Permutations row is $120 = 5!$ and again we have accounted for all permutations. We then calculate the average extent to be $\frac{1\cdot 0+15\cdot 1+38\cdot 2+42\cdot 3+24\cdot 4}{5!} = \frac{313}{120} \approx 2.6083$.

❏

While these tables did show the correct number of permutations for each extent and we were able to calculate the average extents when $n = 1, 2, 3, 4, 5$, they ended up not being particularly helpful in solving the problem by themselves.

5. More Extents

Since I was not seeing any obvious patterns to explain the values in the the Number of Permutation rows of the previous tables, I decided to try to count the number of ways to get the possible extents for a specific case of n without writing out all $n!$ permutations. In order to do this, I made tables with what I called *Position Permutations.*

Definition 5.1 (Position Permutation) *A position permutation of extent e is an arrangement of rings on a tower such that an extent of e will occur.*

Remark 5.2 *When convenient, I will abbreviate "Position Permutation" to "PP".*

It is important to note that any one position permutation could have multiple distinct permutations of rings which each satisfy the same position permutation. This idea will become clearer with specific examples.

Consider the case when $n = 10$. By Lemma 3.1, we know that the largest extent value when $n = 10$ is 9. In order to count the number of permutations which have an extent of 9, I made a table to show the possible position permutations which would result in an extent of 9. Since I want an extent of 9 and I have 10 rings, I have exactly 1 ring to drop that will land in a position on the rod. Since I need the extent to lie above the 10th position and I only have 1 ring to drop on the tower, that ring must fill the 10th position. This means there will be exactly one position permutation for the case where $e = 9$. This is demonstrated in the table below.

	Position Permutation 1
10	x
9	
8	
7	
6	
5	
4	
3	
2	
1	

We see that for PP1, the only ring I can drop that will land in position 10 on the first drop is Ring 10. Then since I have an extent of 9, there are exactly 9! permutations which will result in an extent of 9.

A much more elegant way of showing this, is through the use of Lemma 3.3 and its corollary. By these, we know that the only way to get an extent of 9 is to drop Ring 10 first. We also know there are exactly $(10-1)! = 9!$ permutations which will create this extent value.

When I want to make an extent of 8, my lemmas are not as helpful because they do not allow me to count all of the permutations which result in an extent of 8. In order to count the number of permutations which have an extent of 8 then, I will use my method of creating position permutations.

To form an extent of 8, I have exactly two rings I must place on the tower, and I must place them so that the 2nd ring I drop lands in position 10. For this and subsequent values of e, I will make tables of position permutations where I specify which ring has been dropped first. For this case I have two rings to drop, and thus the highest ring I can drop first is Ring 9.

	Position Permutation 1
10	x
9	x
8	
7	
6	
5	
4	
3	
2	
1	

Since in the case of this position permutation I drop ring 9 first, there is exactly one choice for the first ring I drop. Then since any of the remaining 9 rings will land in position 10, there are 9 rings I may drop next, meaning there are $1 \cdot 9 = 9$ ways to drop those first two rings.
The following tables show what will happen when I drop ring 8 first, ring 7 first, and so on.

	$PP1$
10	x
9	
8	x
7	
6	
5	
4	
3	
2	
1	

	$PP1$
10	x
9	
8	
7	x
6	
5	
4	
3	
2	
1	

	$PP1$
10	x
9	
8	
7	
6	x
5	
4	
3	
2	
1	

	$PP1$
10	x
9	
8	
7	
6	
5	x
4	
3	
2	
1	

	$PP1$
10	x
9	
8	
7	
6	
5	
4	x
3	
2	
1	

	$PP1$
10	x
9	
8	
7	
6	
5	
4	
3	x
2	
1	

	$PP1$
10	x
9	
8	
7	
6	
5	
4	
3	
2	x
1	

	$PP1$
10	x
9	
8	
7	
6	
5	
4	
3	
2	
1	x

We can see from each of these tables that there is exactly one way to drop the first ring, and in fact there is exactly one way to drop the second ring as well since in each table position 9 is not filled and the second ring fills position 10. The only way to fill position 10 when position 9 is not filled is to drop ring 10. Because of this, we see that whenever there is a gap between any two x's in any table there is exactly one way to fill the position of the x directly above the gap.

For each of the eight tables above, there is one way to drop the first ring and one way to drop the second ring, and thus there are a total of eight ways to drop the two rings such that the first ring is any ring from Ring 1 to Ring 8, and the second ring is ring 10.

Since we saw previously that there were nine ways to drop the two rings when we drop ring 9 first, and now we see there are 8 ways to drop the two rings such that the first ring lands below position 9, there is a total of $9 + 8 = 17$ ways to drop two rings such that the 2nd ring dropped lands in position 10. We multiply this 17 by 8! to account for all the ways to permute the eight rings that make up the extent. This gives a total number of permutations where $e = 8$ of $17 \cdot 8!$.

Now we will explore when $e = 7$. In this case, I have 3 rings to place so that the last one lands at position 10. Because of this, the highest possible ring I could drop first is ring 8. Suppose I drop ring 8 first. Then the table of position permutations is the following.

	Position Permutation 1
10	x
9	x
8	x
7	
6	
5	
4	
3	
2	
1	

We see that there is only one way to choose the first ring I drop since I specified that ring 8 was to be dropped first. Then there are eight remaining rings that could fit on position 9 (keep in mind that while there are nine rings remaining to be dropped, one of those rings, Ring 10, could not fall to position 9, and thus of the nine remaining rings, only eight of them could possibly fill position 9). Then there are eight rings remaining to be dropped and since any of them could fill position 10, there are eight ways to fill that position. Thus there are $1 \cdot 8 \cdot 8 = 8^2$ ways to fill this position permutation.

Now suppose we drop ring 7 first. I still need the third ring I drop to land in position 10, and since I have specified that position 7 has been filled, there are two positions, position 8 and position 9, which are to be filled with the remaining ring that is not filling position 7 or position 10. This means there will be $\binom{2}{1} = 2$ position permutations. This is shown below.

	Position Permutation 1	Position Permutation 2
10	x	x
9		x
8	x	
7	x	x
6		
5		
4		
3		
2		
1		

For PP 1, there is one way to drop the first ring, seven ways to drop the second ring (keep in mind that only the rings with indices less than or equal to eight can fill position 8), and one way to drop the third ring. This makes for a total of $1 \cdot 7 \cdot 1 = 7$ ways to fill PP 1.

For PP 2, there is one way to drop the first ring, one way to drop the second ring, and eight ways to drop the third ring. This makes for a total of $1 \cdot 1 \cdot 8 = 8$ ways to fill PP 2.

Combining the values for PP 1 and PP 2 gives a total of $7 + 8 = 15$ ways to fill the position permutations where ring 7 is dropped first.

Now suppose we drop ring 6 first. This time, we have three positions besides positions 6 and 10 to fill and still only one ring to fill those three positions since we must have seven rings above position 10 in order to form the extent of seven. This means there will be $\binom{3}{1} = 3$ position permutations where ring 6 is dropped first. This is shown below.

	Position Permutation 1	Position Permutation 2	Position Permutation 3
10	x	x	x
9			x
8		x	
7	x		
6	x	x	x
5			
4			
3			
2			
1			

Using the previous counting strategies, we see that there are $1 \cdot 6 \cdot 1 = 6$ ways to fill PP 1, $1 \cdot 1 \cdot 1 = 1$ ways to fill PP 2, and $1 \cdot 1 \cdot 8 = 8$ ways to fill PP 3. This gives a total of $6+1+8 = 15$ ways to fill the position permutations where ring 6 is dropped first.

I continued creating tables of position permutations and counting them in this manner for the remaining cases of rings to be dropped first. This is when an interesting pattern arose: for all of the cases where I drop Ring 7, Ring 6, ..., or Ring 1 first, the number of ways to fill those position permutations was always 15. In fact, if we look back to the tables for the $e = 8$ case, we see that each of the cases where Ring 8, Ring 7, ..., or Ring 1 is dropped first had the same number of ways to satisfy the position permutations (1). At this point, it looks like we will get the same number of ways to fill the position permutations regardless of if we drop Ring e, Ring $e-1$, ..., or Ring 1 first.We will examine further values of e to see if this pattern continues. Before we do that, however, we want to count the number of permutations which have an extent of 7. We saw there were 8^2 ways to fill the position permutations when we drop Ring 8 first, and there were 15 ways to fill the position permutations when we drop rings 7, 6, down to Ring 1 first. Thus there are $8^2 + 7 \cdot 15 = 169$ ways to fill the position permutations so the extent is 7. We must then multiply by 7! to account for the number of ways to permute the extent, resulting in $169 \cdot 7!$ permutations which have an extent of 7.

Suppose $e = 6$. Then we have four rings to drop on the rod such that the last ring lies at the 10th position. This means I can drop rings 7, 6, down to 1 first and still be able to get an extent of 6. Suppose I drop Ring 7 first. The table below shows the position permutation that illustrates this.

	Position Permutation 1
10	x
9	x
8	x
7	x
6	
5	
4	
3	
2	
1	

Using the same counting methods as above, we see that there is one way to drop the first ring, and 7 ways each to drop the remaining 3 rings. Thus for this position permutation there are $1 \cdot 7 \cdot 7 \cdot 7 = 7^3$ ways to fill it.

Now suppose I drop Ring 6 first. Positions 6 and 10 must be filled, which leaves two rings to fill the three remaining positions, positions 7, 8, and 9. This results in the $\binom{3}{2} = 3$ position permutations shown below.

	Position Permutation 1	Position Permutation 2	Position Permutation 3
10	x	x	x
9		x	x
8	x		x
7	x	x	
6	x	x	x
5			
4			
3			
2			
1			

Using the same counting methods as before, we see there are $1 \cdot 6 \cdot 6 \cdot 1 = 36$ ways to fill PP1, $1 \cdot 6 \cdot 1 \cdot 7 = 42$ ways to fill PP2, and $1 \cdot 1 \cdot 7 \cdot 7 = 49$ ways to fill PP3. This makes for a total of $36 + 42 + 49 = 127$ ways to fill the position permutations when Ring 6 is dropped first.

Now we predict that the total number of ways to fill the position permutations when Ring 5 is dropped first is 127 as well. The following will verify that is true. Suppose I drop Ring 5 first. Using the same counting method as before, there are $\binom{4}{2} = 6$ position permutations, shown below.

	$PP1$	$PP2$	$PP3$	$PP4$	$PP5$	$PP6$
10	x	x	x	x	x	x
9			x		x	x
8		x	x	x		
7	x	x			x	
6	x			x		x
5	x	x	x	x	x	x
4						
3						
2						
1						

Again using the same counting methods, we get $1 \cdot 5 \cdot 5 \cdot 1 = 25$ ways to fill PP 1, $1 \cdot 1 \cdot 6 \cdot 1 = 6$ ways to fill PP 2, $1 \cdot 1 \cdot 7 \cdot 7 = 49$ ways to fill PP 3, $1 \cdot 5 \cdot 1 \cdot 1 = 5$ ways to fill PP 4, $1 \cdot 1 \cdot 1 \cdot 7 = 7$ ways to fill PP 5, and $1 \cdot 5 \cdot 1 \cdot 7 = 35$ ways to fill PP 6. This gives a total of $25+6+49+5+7+35=127$ ways to fill the position permutations when Ring 5 is dropped first.

It appears that the pattern is indeed continuing, but the only way to know for sure is to check the remaining cases for when we drop rings 4, 3, 2, and 1 first. I did those tables by hand and sure enough, the pattern did continue. This means the total number of ways to fill the position permutations when $e=6$ is $7^3+127\cdot 6=1105$. Then in order to account for the number of ways to permute the extent, we multiply by 6! to get $1105\cdot 6!$ permutations with an extent of 6.

I continued in this manner, calculating the number of position permutations with extents of 5, 4, 3, 2, and 1. These can be seen in the table below, along with the previously calculated values for extents of 6, 7, 8, and 9.

	Number of Ways to fill PP's
$e=9$	1
$e=8$	$9+8\cdot 1$
$e=7$	$8^2+7\cdot 15$
$e=6$	$7^3+6\cdot 127$
$e=5$	$6^4+5\cdot 671$
$e=4$	$5^5+4\cdot 2101$
$e=3$	$4^6+3\cdot 3367$
$e=2$	$3^7+2\cdot 2059$
$e=1$	$2^8+1\cdot 255$

At this point I was sure there was some kind of pattern, but I was not sure exactly what it looked like. The first thing I noticed was that $15=8^2-7^2$, and from there I realized $127=7^3-6^3$. At this point I thought maybe I had figured out the pattern so I followed through the table with what the pattern would suggest, and sure enough, the pattern I saw was accurate.

	Number of Ways to fill PP's	Patterned Numer of Ways to fill PP's
$e=9$	1	$10^0+9\cdot(10^0-9^0)$
$e=8$	$9+8\cdot 1$	$9^1+8\cdot(9^1-8^1)$
$e=7$	$8^2+7\cdot 15$	$8^2+7\cdot(8^2-7^2)$
$e=6$	$7^3+6\cdot 127$	$7^3+6\cdot(7^3-6^3)$
$e=5$	$6^4+5\cdot 671$	$6^4+5\cdot(6^4-5^4)$
$e=4$	$5^5+4\cdot 2101$	$5^5+4\cdot(5^5-4^5)$
$e=3$	$4^6+3\cdot 3367$	$4^6+3\cdot(4^6-3^6)$
$e=2$	$3^7+2\cdot 2059$	$3^7+2\cdot(3^7-2^7)$
$e=1$	$2^8+1\cdot 255$	$2^8+1\cdot(2^8-1^8)$

Based on this, I came up with a conjecture for a general formula to count the number of position permutations for each extent e using the variables e and n.
I then noticed a pattern in the number of ways to fill the position permutations that relied on e and n. The formula I came up with was

$$(e+1)^{n-(e+1)} + e\Big((e+1)^{n-(e+1)} - e^{n-(e+1)}\Big).$$

When I simplified this, something really exciting happened.

$$\begin{aligned}(e+1)^{n-(e+1)} + e\Big((e+1)^{n-(e+1)} - e^{n-(e+1)}\Big) &= (e+1)^{n-(e+1)} + e(e+1)^{n-(e+1)} - e^{n-e} \\ &= (e+1)^{n-(e+1)}(1+e) - e^{n-e} \\ &= (e+1)^{n-e} - e^{n-e}\end{aligned}$$

Note that this formula is counting the number of ways to get a certain extent e. When we have an extent of e, we have dropped $n-e$ rings onto the tower of height n such that the last ring lies at position n. We can denote the $n-e$ rings we drop as m rings and then we have $n-e=m$ and also $e=n-m$. Substituting these values in the previous equation gives $(n-m+1)^m - (n-m)^m$. Therefore by transitivity,

$$(e+1)^{n-(e+1)} + e\Big((e+1)^{n-(e+1)} - e^{n-(e+1)}\Big) = (n-m+1)^m - (n-m)^m.$$

We then get that this is the number of ways to drop m rings onto a tower of height n such that the last ring lies at position n. This was exciting because this is exactly the solution Bonomo and Cuff found in their paper for this problem. This is the point where my research lined up with Bonomo's and Cuff's work.

6. Solving the problem

Recall that the initial problem asked *What is the average extent of all permutations of n rings?* When I began my research, I focused on individual permutations which in the end was not a successful method. Then I focused on counting ways to get any given extent. This method was more successful and turned out to be more in line with what Bonomo and Cuff did. In their paper, they realized that before they could solve the original problem, there were smaller problems that when solved first would provided a means to solve the original problem. In their paper, they came up with two of these additional problems that they then used to solve the original question.

6.1. Additional Problems

In [**1**], Bonomo and Cuff described the problems as follows:
Problem 0. Determine the average extent of all permutations of the n rings.
Problem 1. Determine the number of ways to drop m rings onto a tower of height n such that the last ring lies at the nth location.
Problem 2. Determine the number of ways to drop m rings onto a tower of height n such that the last ring does not lie above the nth location.
Their strategy was to begin by solving Problem 2 and work their way back up to Problem 1.

6.2. Problem 2

Bonomo and Cuff begin by letting $T(n, m)$ denote the number of ways to drop m rings onto a tower of height n such that the last ring does not lie above the nth location. Thus finding a solution for $T(n, m)$ solves Problem 2.

Through their work, they found a recurrence relation to solve this problem. They explained their recurrence relation in the following manner.

Suppose we know the value of $T(n-1, m-1)$, meaning we know the number of ways to drop $m-1$ rings onto a tower of height $n-1$ such that the last ring we drop does not lie above position $n-1$. Then suppose we allow the tower to grow by a value of 1, meaning we now have a tower of size n. Since we are working with a tower of size n, we can drop an additional ring onto the $m-1$ rings we have already dropped without worrying that the added ring will exceed position n, thereby becoming part of the extent. Now we just have to count the number of ways we can choose that extra ring we drop. We have n rings total, but we have already dropped $m-1$ of them, so we have $n-(m-1) = n-m+1$ rings to choose from when we pick that extra ring to drop. This means that for every configuration of $T(n-1, m-1)$, we have $n-m+1$ rings to choose from to make the $T(n, m)$. Thus the recurrence relation is $T(n, m) = (n-m+1)T(n-1, m-1)$.

The authors then note that when $m = 1$, $T(n, m) = n$ since there are exactly n ways to drop $m = 1$ of n rings onto a tower of size n. They simultaneously note that $m > n$ is not possible because we cannot have more rings to drop than the size of the tower. Thus we must have $m \leq n$.

They state the recurrence relation for all possible values of m:

$$T(n,m) := \begin{cases} n & m = 1 \\ (n-m+1)T(n-1,m-1) & 1 < m \leq n. \end{cases} \tag{6.1}$$

At this point, they used a computer program to generate a table of values for $T(n,m)$ for various values of n and m. From this, they proposed a closed form for $T(n,m)$. This was $T(n,m) = (n-m+1)^m$. They proceeded by proving this equation by Induction. The following is the proof they gave in [**1**] with minor stylistic adjustments.

Theorem: If $T(n,m)$ denotes the number of ways to drop m rings onto a tower of height n such that the last ring does not lie above the nth location, then $T(n,m) = (n-m-1)^m$.

Proof: For the Base Case, we let $n = 1$. Since $m \leq n$, we see that $m = 1$ is the only valid choice for the value of m. We want to show $T(1,1) = (1-1+1^1) = 1$. Recall by the definition of $T(n,m)$, that $T(1,1)$ is the number of ways to drop 1 ring onto a rod of size 1 such that the highest ring does not lie above the 1st position. There is clearly exactly 1 way to drop 1 ring onto a rod of size 1, and thus our base case holds.

Next, we form the Induction Hypothesis. For this step, we assume for some value $k-1$, $T(k-1,m) = ((k-1)-m+1)^m$ holds for $1 \leq m \leq k-1$.

For the Induction Step, we look at the value $(k-1)+1 = k$. We have two cases to check: when $m = 1$, and when $1 < m \leq (k-1)+1 = k$. Suppose $m = 1$. Then $T(k,1) = k$ by equation 6.1. Also, we note that $T(k,1) = k = (k-1+1)^1$, and so the formula holds when $m = 1$.

Now suppose $1 < m \leq k$. By equation 6.1,

$$\begin{aligned} T(k,m) &= (k-m-1)T(k-1,m-1) \\ &= (k-m-1)((k-1)-(m-1)+1)^{m-1} \quad \text{(by the Induction Hypothesis)} \\ &= (k-m-1)(k-m+1)^{m-1} \\ &= (k-m-1)^m \end{aligned}$$

as desired. Thus by induction, $T(n,m) = (n-m+1)^m$.

❏

With this, the authors have solved Problem 2.

6.3. Problem 1

After solving Problem 2, the solution for Problem 1 follows in a very straightforward manner. Problem 1 stated: Determine the number of ways to drop m rings onto a tower of height n such that the last ring lies at the nth location. Since Problem 2 counts the number of ways to drop m rings onto a tower of height n such that the last ring lies at the nth location or below, all we have to do is subtract out the number of permutations in which the mth ring dropped lies at or below the $(n-1)$th position.

This means if we let $S(n, m)$ denote the number of ways to drop m rings such that the last ring lies at position n, then $S(n, m) = T(n, m) - T(n-1, m)$. By the solution to Problem 2, we see that

$$\begin{aligned} S(n, m) &= T(n, m) - T(n-1, m) \\ &= (n - m + 1)^m - ((n-1) - m + 1)^m \\ &= (n - m + 1)^m - (n - m)^m \end{aligned}$$

And with that, Problem 1 is solved. We see here this is the same formula that I predicted when calculating the extents of the $n = 10$ case. By Bonomo and Cuff's work, we can see that my formula is correct.

We can now use the solution to Problem 1 to find the solution to our original problem, Problem 0.

First, Bonomo and Cuff note one way we can check our formula. Since $S(n, m)$ counts the number of ways to drop m rings such that the last ring lands in position n, we know that for each of those ways, there are $(n - m)!$ ways to permute the produced extent of size $n - m$. Thus for a tower of height n, there are $S(n, m)(n - m)!$ permutations with an extent of $(n - m)$. This means that if we sum these permutations accounting for extent sizes of 0 to $n - 1$ we should get the number of permutations of n rings, $n!$, since for any extent e, $0 \leq e \leq n - 1$. Thus we want to check that

$$\sum_{m=1}^{n} S(n, m)(n - m)! = n!$$

The following is the proof Bonomo and Cuff give in [**1**], with a few additional steps.

$$
\begin{aligned}
\sum_{m=1}^{n} S(n,m)(n-m)! &= \sum_{m=1}^{n} \Big((n-m+1)^m - (n-m)^m \Big)(n-m)! \\
&= \sum_{m=1}^{n} \Big((n-m-1)^m (n-m)! - (n-m)^m (n-m)! \Big) \\
&= \sum_{m=1}^{n} (n-m-1)^m (n-m)! - \sum_{m=1}^{n} (n-m)^m (n-m)! \\
&= \sum_{m=0}^{n-1} \Big(n-(m+1)+1 \Big)^{m+1} \Big(n-(m+1) \Big)! - \sum_{m=1}^{n} (n-m)^m (n-m)! \\
&= \sum_{m=0}^{n-1} (n-m)^{m+1} (n-m-1)! - \sum_{m=1}^{n} (n-m)^m (n-m)! \\
&= \sum_{m=0}^{n-1} (n-m)^{m+1} (n-m-1)! - \Big((n-n)^n (n-n)! + \sum_{m=1}^{n-1} (n-m)^m (n-m)! \Big) \\
&= \sum_{m=0}^{n-1} (n-m)^{m+1} (n-m-1)! - \Big(0 + \sum_{m=1}^{n-1} (n-m)^m (n-m)! \Big) \\
&= \sum_{m=0}^{n-1} (n-m)^{m+1} (n-m-1)! - \sum_{m=1}^{n-1} (n-m)^m (n-m)! \\
&= (n-0)^{0+1}(n-0-1)! + \sum_{m=1}^{n-1} (n-m)^{m+1} (n-m-1)! - \sum_{m=1}^{n-1} (n-m)^m (n-m)! \\
&= n(n-1)! + \sum_{m=1}^{n-1} \Big((n-m)^{m+1} (n-m-1)! - (n-m)^m (n-m)! \Big) \\
&= n! + \sum_{m=1}^{n-1} (n-m)^m (n-m-1)! \Big((n-m) - (n-m) \Big) \\
&= n! + \sum_{m=1}^{n-1} 0 \\
&= n!
\end{aligned}
$$

Thus $\displaystyle\sum_{m=1}^{n} S(n,m)(n-m)! = n!$.

This shows that we have accounted for all permutations. From here, the solution to Problem 0 is fairly straightforward.

6.4. Problem 0

Recall that Problem 0 asked us to find the average extent of n rings. This means we need to sum the extents of all permutations of the n rings, and then divide by the total number of permutations, $n!$. In the previous subsection we showed that $\sum_{m=1}^{n} S(n,m)(n-m)!$ accounts for all of the permutations. Each of these permutations contributes an extent of $n-m$, and so we can easily see that $\sum_{m=1}^{n}(n-m)S(n,m)(n-m)!$ will count the sum of the extents of all permutations of the n rings. Thus, our last step is to divide by $n!$ to calculate the average extent of n rings and solve the problem. Bonomo and Cuff denote the solution as

$$H(n) = \frac{1}{n!}\sum_{m=1}^{n}(n-m)S(n,m)(n-m)!$$

and simplify this. The following is the solution they give in [**1**] along with some additional steps.

$$
\begin{aligned}
H(n) &= \frac{1}{n!}\sum_{m=1}^{n}(n-m)S(n,m)(n-m)! \\
&= \frac{1}{n!}\sum_{m=1}^{n}(n-m)\Big((n-m+1)^m-(n-m)^m\Big)(n-m)! \\
&= \frac{1}{n!}\left[\sum_{m=1}^{n}\Big((n-m)(n-m)!(n-m+1)^m-(n-m)(n-m)!(n-m)^m\Big)\right] \\
&= \frac{1}{n!}\Bigg(\sum_{m=1}^{n}(n-m)(n-m)!(n-m+1)^m-\sum_{m=1}^{n}(n-m)^{m+1}(n-m)!\Bigg) \\
&= \frac{1}{n!}\Bigg(\sum_{m=0}^{n-1}(n-(m+1))(n-(m+1))!(n-(m+1)+1)^{m+1}-\sum_{m=1}^{n}(n-m)^{m+1}(n-m)!\Bigg) \\
&= \frac{1}{n!}\Bigg(\sum_{m=0}^{n-1}(n-m-1)(n-m-1)!(n-m)^{m+1}-\sum_{m=1}^{n}(n-m)^{m+1}(n-m)!\Bigg) \\
&= \frac{1}{n!}\Bigg((n-0-1)(n-0-1)!(n-0)^{0+1}+\sum_{m=1}^{n-1}(n-m-1)(n-m-1)!(n-m)^{m+1} \\
&\qquad -\sum_{m=1}^{n}(n-m)^{m+1}(n-m)!\Bigg) \\
&= \frac{1}{n!}\Bigg((n-1)(n-1)!n+\sum_{m=1}^{n-1}(n-m-1)(n-m-1)!(n-m)^{m+1}-\sum_{m=1}^{n-1}(n-m)^{m+1}(n-m)!\Bigg) \\
&= \frac{1}{n!}\left[(n-1)(n-1)!n+\sum_{m=1}^{n-1}\Big((n-m-1)(n-m-1)!(n-m)^{m+1}-(n-m)^{m+1}(n-m)!\Big)\right] \\
&= (n-1)+\frac{1}{n!}\left[\sum_{m=1}^{n-1}\Big((n-m-1)(n-m-1)!(n-m)^{m+1}-(n-m)^{m+1}(n-m)!\Big)\right] \\
&= (n-1)+\frac{1}{n!}\Bigg(\sum_{m=1}^{n-1}(n-m)^{m+1}(n-m-1)!((n-m-1)-(n-m))\Bigg) \\
&= (n-1)+\frac{1}{n!}\Bigg(\sum_{m=1}^{n-1}(n-m)^{m+1}(n-m-1)!(-1)\Bigg) \\
&= (n-1)-\frac{1}{n!}\Bigg(\sum_{m=1}^{n-1}(n-m)^{m+1}(n-m-1)!\Bigg) \\
&= (n-1)-\frac{1}{n!}\Bigg(\sum_{m=1}^{n-1}(n-m)^{m}(n-m)!\Bigg) \\
&= (n-1)-\frac{1}{n!}\Bigg(\sum_{m=1}^{n-1}m^{n-m}m!\Bigg)
\end{aligned}
$$

To justify the last step, we use commutativity of addition shown below:

$$\begin{aligned}\sum_{m=1}^{n-1}(n-m)^m(n-m)! &= (n-1)^1(n-1)! + (n-2)^2(n-2)! + \cdots + (n-(n-2))^{n-2}(n-(n-2))! \\ &\quad + (n-(n-1))^{n-1}(n-(n-1))! \\ &= (n-1)^1(n-1)! + (n-2)^2(n-2)! + \cdots + (2)^{n-2}(2)! + (1)^{n-1}(1)! \\ &= (1)^{n-1}(1)! + (2)^{n-2}(2)! + \cdots + (n-2)^2(n-2)! + (n-1)^1(n-1)! \\ &= (1)^{n-1}(1)! + (2)^{n-2}(2)! + \cdots + (n-2)^{n-(n-2)}(n-2)! + (n-1)^{n-(n-1)}(n-1)! \\ &= \sum_{m=1}^{n-1} m^{n-m}m!\end{aligned}$$

This gives us the final solution

$$H(n) = (n-1) - \frac{1}{n!}\left(\sum_{m=1}^{n-1} m^{n-m}m!\right)$$

With this, we have a solution for Problem 0. Unfortunately, Bonomo and Cuff note that they are not able to derive a closed form solution from this equation. However, with this formula it is theoretically possible to find the average extent of n rings on a stacking ring tower, given we have either a lot of time on our hands, or a very powerful computer.

Recall that in Lemmas 4.1-4.5 I manually calculated the average extent of n rings for the cases of $n = 1$, $n = 2$, $n = 3$, $n = 4$, and $n = 5$. Using Bonomo and Cuff's formula and a simple graphing calculator, I was able to verify that the values their formula gives match the values I calculated for those cases. The following table shows this.

n	My Lemma Value	Bonomo and Cuff's Formula Value: $H(n)$
1	$\frac{0}{1} = 0$	0
2	$\frac{1}{2} = 0.5$	0.5
3	$\frac{7}{6} \approx 1.167$	≈ 1.167
4	$\frac{45}{24} = 1.875$	1.875
5	$\frac{313}{120} \approx 2.6083$	≈ 2.6083

And with that, we see that the values I calculated correspond to Bonomo and Cuff's formula.

7. Conclusion

After solving the original problem in their paper [**1**], Bonomo and Cuff went on to tackle more problems involving a stacking ring tower. These included the likely event of the existence of duplicate rings and/or missing rings. For more information about their conclusions on this work, see [**1**].

The Primary problem presented within John P. Bonomo and Carolyn K. Cuff's paper *How Do You Stack Up?* was to find a way to count the average extent of n rings on a stacking ring tower. In my own attempt to solve this problem, I tried multiple approaches, one of which led me to an answer to one of the smaller problems Bonomo and Cuff proved in their paper. While I was not able to prove my method on my own, I was able to form a conjecture and validate it using Bonomo and Cuff's work. I was able to examine Bonomo and Cuff's work and show where and explain why their solution matched the one I predicted. In the end, I was able to thoroughly understand their solution and their methodology behind it. Although our approaches were slightly different, both provided the means to solve the original problem.

REFERENCES

1. J.P. Bonomo and C.K. Cuff, How do you stack up? *The College Mathematics Journal* **35** (2004) 351–361.